This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

3 3433 06644293 4

A TREATISE

ON

OPTICS

BY

S. PARKINSON, B.D.

FELLOW AND TUTOR OF ST JOHN'S COLLEGE, CAMBRIDGE.

ASTOR LIBRARY
NEW-YORK.

MACMILLAN AND CO.
Cambridge :
AND 23, HENRIETTA STREET, COVENT GARDEN, LONDON.
1859.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

PREFACE.

THE present work may be regarded as a new edition of the *Treatise on Optics* by the Rev. W. N. Griffin, which being some time ago out of print, was very kindly and liberally placed at my disposal by the Author. I have freely used the liberty accorded to me, and have rearranged the matter with considerable alterations and additions—especially in those parts which required more copious explanation and illustration to render the work suitable for the present course of reading in the University. The numerous diagrams which the subject requires have been inserted in the body of the work, instead of being collected in plates at the end, and are thus rendered more convenient for reference.

I have appended a collection of examples and problems—distributed under the heads of the several chapters, as well as a miscellaneous set,—which are sufficiently numerous and varied in character to afford a useful exercise for the student: for the greater part of them I have had recourse to the

Examination Papers set in the University and the several Colleges during the last twenty years.

Subjoined to the copious Table of Contents I have ventured to indicate an elementary course of reading, not unsuitable for the requirements of the present examination of the *First Three Days* in the Senate-House.

S. PARKINSON.

ST JOHN'S COLLEGE,
October, 1859.

CONTENTS.

CHAPTER I.

PAGE

Laws of Propagation of Light:—Direct Reflexion and Refraction 1

Subject of this treatise, *Geometrical* or *Formal Optics*, Art. 1; Definition of *self-luminous, medium, rays, pencil* of light, incidence *direct* and *oblique*, 2—7; *laws of reflexion* and *refraction*, 8—10; foundation of such laws, 11; Def. of *Geometrical* and *Principal Focus, focal length, aberration*, 12, 13; *Reflexion* and *refraction* at a *plane surface*, 14—19; Discussion of the *Geometrical focus* and *conjugate foci* of a pencil directly reflected at a spherical surface, 21—25; *Geometrical* and *visible image*, 26; Discussion of a pencil directly refracted at a spherical surface, 27—31; Illustrations,—successive reflexions at *parallel mirrors* and *inclined mirrors*, 32—35; *aplanatic surfaces*, 36.

CHAPTER II.

Illumination of Surfaces . . . 29

Measure and *law* of illumination, 37—40; law of *intensity of emission*, 41; method of calculating illumination, 42, 43; objects appear equally bright at all distances, 44; Ritchie's *photometer*, 45; example, 46; velocity of light, 47.

CHAPTER III.

Aberration of small direct pencils . . 37

Measure of aberration, 48; Formula for aberration after *refraction at a plane surface*, 50, 51; after *reflexion at a spherical surface*, 52; after *refraction at a spherical surface*, 53; *Least circle of aberration*, its formation, position and dimensions, 55, 56.

CHAPTER IV.

PAGE
Focal Lines of small oblique pencils,—Caustics . 48

Astigmatism, caustic curve, 57; mode of determining the form of a caustic, —*catacaustics, diacaustics,* 58; Def. of *centres* of the *surface* and *face,*— incidence *centrical* and *excentrical*—*axis* of a surface, 59; *oblique* pencil reflected at a plane surface, 60; Formation of *focal lines* by an oblique pencil,—Def. of *Primary* and *Secondary focal line,*—*circle of least confusion,* 62—64; Illustration of *geometrical* and *visible image,* 65; Position of the Foci of a small oblique pencil after *reflexion* at a spherical surface, 66; the same after *refraction* at a plane or spherical surface, 67—69; Position and dimensions of the focal lines and circle of least confusion, 70; Illustrations,—caustic after reflexion at a circle in certain cases, 71, 72; caustic after refraction at a plane surface, 73, 74.

CHAPTER V.

Successive reflexions and refractions;—prisms—lenses . 68

Successive reflexions at plane surfaces, 76—78; successive refractions at plane surfaces,—*critical angle,* 79—81; *a plate,* 82; *absolute* and *relative index* of refraction, 83, 84; refraction through a plate, 85—87; Def. of *edge, faces, refracting angle, principal section of a prism,* 88; *deviation* through a prism, *minimum* deviation, 90—94; foci of a small pencil passing through a prism, 96; Def. of a *lens, axis* of a lens, *forms* of lenses, 97, 98; geometrical focus of a pencil passing through a lens, 99, 100; *principal focus, focal length,* of a lens,—*concave* and *convex* lens,—*power* of a lens, 101—104; a *sphere* regarded as a lens,— its *focal* length, 105, 106; relative motion of conjugate foci in the case of a lens, 107; focal length of a lens determined *practically,* 108; *centre* of a lens, 109; *focal centres* of a lens, 110; Discussion of an oblique pencil passing through a lens, 112, 113; combinations of lenses, 114; approximate determination of the focus of a small oblique pencil, 115; Lenses used in Lighthouses, 117.

CHAPTER VI.

Refraction through media of varying density;—reflexion and refraction at a surface in any manner . . . 102

Absolute and *relative* refractive index, 118; method of determining the path of a ray in a medium of varying refractive power under various circumstances, 120—125; Formulæ for determining the course of a ray reflected or refracted at two or more surfaces in any manner, 126, 127.

CHAPTER VII.

Spherical aberration of lenses;—Excentrical pencils . 114

Aberration of a direct pencil, 129; form of a lens to produce *minimum aberration* for rays parallel,—or not so, 130—132; *examples,* 133; Excentrical pencil passing through one or more lenses—mode of defining it in simple cases, 134—138; *equivalent* lens, 139; examples, Ramsden's and Huyghens' eye-pieces, 140.

CHAPTER VIII.

Images and Caustics . . . 126

Image—*geometrical* or *visible,—real* or *virtual,—erect* or *inverted,* 141—143; Example, image of a straight line formed by a lens, 144—146; *Distortion,—linear, angular* and that of *curvature,* 147; Deformation of an image relatively to the object, 148, 149; calculation of the distortion in the case of a spherical mirror, 150; Brightness of an image, 151, 152; length, &c. of a caustic, 153, 154.

CHAPTER IX.

On the Chromatic Dispersion of Light . 142

Newton's experiment, 156; *pure spectrum,* how observed by Wollaston and Fraunhofer, 157, 158; *fixed lines* of the spectrum, 159—161; mode of determining the *refracting angle* of a prism,—the *minimum deviation* of a given colour—and the *index of refraction,—test of the law of refraction,* 162—165; *standard scale* of colours, 166; Def. of *deviation, dispersion, irrationality of dispersion, secondary spectra,* 167; dispersion of two colours by a prism, 168; dispersive power, 171; *achromatic combinations,—possible* but *imperfect,* 172, 173; conditions of achromatism for *prisms,* 174—176; for *lenses,* 177—181; *circle of chromatic aberration,* 182—184; colours of natural bodies, primary colours, 185.

CHAPTER X.

PAGE

Of Vision and Optical Instruments . . 169

The Eye,—retina, crystalline lens, uvea, punctum cæcum, 187—190; *Binocular* vision, 191; *Pseudoscope,* 192; dimensions, &c. of the eye, 194; Defects of vision, *long* and *short sight,* 195; vision through a lens, 197; *visual angle* of an object seen through a lens, 198; *magnifying power* of a lens, 200, 201; Telescopes, their object, 203; the *Astronomical* Telescope, its *field of view,* and *magnifying power,—the ragged edge and stop,* 204—206; *Galileo's* Telescope, 207—210; *Newton's* Telescope, 211—213; *Herschel's* Telescope, 214—216; *Gregory's* and *Cassegrain's* Telescopes, 217—221; Defects of telescopes, 223—225; *specula* for reflecting telescopes, 226—228; the *finder* of a telescope, 229; object-glasses, 230; Eye-pieces, 231, 232; Astronomical telescope with Huyghens' or Ramsden's Eye-piece, 233—236; the erecting eye-piece, 237; *cross-wires* of an eye-piece, 239; *Dynamometer,* 241—243; *Microscopes,* 244; *simple* and *compound,* 245—248; magnifying power of a microscope, 249; Wollaston's *microscopic doublet,* 246; *the Camera Obscura,* 250; *the Camera Lucida,* 252, 253; *Hadley's Sextant,* 254—256; *the Reflecting Goniometer,* 257.

CHAPTER XI.

Of the Rainbow, &c. . . . 217

Direct refraction through a sphere of water with *one* or *two* internal reflexions, 259; discussion of the caustic formed by parallel rays incident upon a sphere of water and emergent after one internal reflexion, 260, 261; explanation of the *Primary* Rainbow, 262; the caustic after two internal reflexions within a sphere of water, 264—266; the Second Rainbow, 267; order of colours and dimensions of the rainbow, 269—272; general remarks, 273.

* * * * * * *

For an Elementary Course the attention of the Student may be confined to the following articles:—

1—34, 37—41, 44, 45, 57, 58, 65, 74, 76—86, 88—92, 94, 95, 97—111, 114—117, 138 Def., 139—143, 147—149, 186—222, 233—240, 244—252, 254.

OPTICS.

CHAPTER I.

LAWS OF PROPAGATION OF LIGHT:—DIRECT REFLEXION AND REFRACTION.

1. WHEN a material object is presented before us, we become by vision sensible of its existence and figure. In such a case light is said to be propagated from the object to our eyes, and the science of *Optics* has for its design the examination of the circumstances of such propagation.

The science is divided into *Geometrical* and *Physical Optics*. In *Geometrical Optics* the circumstances of the transmission of light are computed on certain laws established by experiment; in *Physical Optics* these laws are accounted for on hypotheses of the structure of bodies, and of the matter filling the space in which they are placed. In a similar manner, in geometrical or plane astronomy the phenomena of heavenly bodies are calculated on observed laws which their apparent motions are found to obey; in physical astronomy these apparent laws are shewn to result from the hypothesis of gravitation.

The former branch of the science is the subject of the present treatise, wherein from certain laws established by experiment under simple circumstances the course of light under more complex circumstances is computed, and the results applied to the construction of optical instruments. These investigations will be conducted in independence of the physical branch of the subject,—the experimental laws on which we commence being equally true, whatever be the nature of the hypothesis which professes to account for them.

2. *Def.* A body is said to be *self-luminous* when it is capable in itself of making our eyes sensible of its existence.

Thus the sun, the stars, a lamp, red-hot iron, &c., are self-luminous, but by far the greater number of natural bodies possess no such property. Such bodies are luminous only by reflexion and require the presence of another luminous body to render them visible. Thus when a lamp is brought into a dark room, the other bodies in the room become visible, and become more or less capable of illuminating others; but this property ceases and they become invisible when the lamp is extinguished or withdrawn.

OBS. When a luminous point *or* an origin of light is mentioned, we may understand it to be a minute portion of a luminous surface, in no direction perceptibly extended: and so by a luminous line we may understand such a surface perceptibly extended in one direction only. We shall regard them simply as a *mathematical point* and a *mathematical line* respectively.

3. *Def.* Any space or substance which light can traverse is called a *medium.* Such as *a vacuum, glass, water,* &c.

4. *Def.* When light emanates from a luminous point, we regard it as made up of *rays*, understanding by a *ray* the smallest portion of light which can be separately transmitted, stopped, or reflected; and we shall treat such rays as mathematical lines.

In a uniform medium, it will be assumed that the course of a ray of light is a straight line; this law is shewn by experiment to be true in general. There are certain cases of apparent exception, such as are presented by the phenomena of diffraction, the explanation of which belongs to physical optics.

5. *Def.* An assemblage of rays proceeding from a luminous point, is a *pencil* of light.

The *form* of a pencil, unless an exception be expressly mentioned, will be regarded as a *right cone* having the origin of light for the vertex, and when this origin is infinitely distant, this cone becomes a circular cylinder as its limiting form. The geometrical axis of the cone or cylinder is called the *axis* of the pencil.

If the rays of a pencil of light produced in a direction opposite to that of propagation meet in a point, the pencil is *divergent;* if the origin be infinitely distant, its limiting form is a pencil of *parallel* rays; if the rays produced in direction of propagation meet in a point, the pencil is *convergent.*

The degree of divergence or convergence may be measured by the vertical angle of the cone which the pencil forms.

6. *Def.* When a pencil meets the surface of any substance or medium, the incidence is called *direct*, if the axis of the pencil coincides with the normal to the surface at the point of incidence: in other cases the incidence is called *oblique.*

7. When light is incident on the surface of a medium different from that in which it is proceeding, a portion of it is *scattered* or *dispersed* over the surface and makes the surface visible; another portion is in general reflected in the first medium (according to a law which will presently be stated)—and in certain cases (as when the medium upon which it is incident is translucent) a third portion enters this new medium according to another law, and is said to be refracted.

The course of the reflected and refracted rays where both exist, may be considered separately.

8. *Law of reflexion.*

When a ray is reflected at the surface of a medium,

(i) The incident and reflected rays lie in one and the same plane with the normal to the surface at the point of incidence, and on opposite sides of the normal.

(ii) The angles which the incident and reflected rays make with the normal to the surface at the point of incidence are equal.

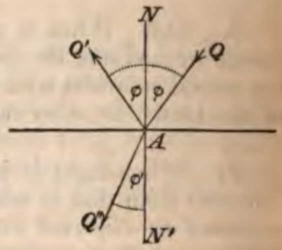
9. *Law of Refraction.*

When a ray is refracted at the surface of a medium,

(i) The incident and refracted rays lie in one and the same plane with the normal to the surface at the point of incidence and on opposite sides of it.

(ii) The sines of the angles which the incident and refracted rays make with the normal to the surface at the point of incidence, have a ratio depending only on the media between which the refraction takes place, and the nature of the light.

10. Thus if a ray QA be incident at A on the surface of a medium (reflecting or refracting or both): NAN' the normal to the surface at A; AQ', AQ'' the reflected and refracted rays respectively; then will the lines QA, $Q'A$, $Q''A$, NAN' all lie in one plane, and the $\angle\, QAN = \angle\, Q'AN$. Also, if we write $\angle\, QAN = \angle\, \phi$, $Q''AN' = \phi'$, then will the ratio $\frac{\sin\phi}{\sin\phi'}$ be constant at whatever angle of incidence a ray of the same kind of light is refracted from one given medium into another.



This constant quantity $\frac{\sin\phi}{\sin\phi'}$, is usually denoted by the symbol μ, and is called the refractive index between the two media for the particular species of light considered. It is a *parameter* which varies, (i) if the nature of the light be altered, (ii) if the relation between the two media be altered.

The relation between ϕ and ϕ' is commonly written

$$\sin\phi = \mu \sin\phi'.$$

It will *in general* be supposed in these pages that the refraction takes place into a denser medium, in which case μ is greater than unity, and the angle of refraction less than the angle of incidence {i. e. $\phi' < \phi$}.

11. *Remark.*

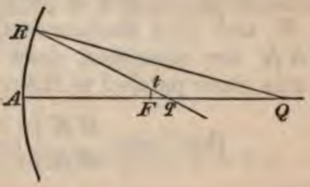
The process by which these and physical laws in general are established experimentally is this. Direct experiments render the law probable; such experiments however are seldom made with such minute accuracy as to prove the law exactly true;—next on supposition of the truth of the law in question, the circumstances of more complex phenomena are computed, and when the results of these computations are found in repeated instances of various kinds to agree minutely with observations, we have a very high degree of probability of the strict truth of the law. It is important to observe that of the truth of the laws of physical science we have only a moral certainty,—a certainty arising only from the improbability that an untrue principle should happen to explain successfully a great variety of phenomena. The first law of motion in Dynamics, for example, we think probably true from experiments made on bodies on the Earth; it is proved by the agreement of the motion of the heavenly bodies calculated on supposition of its truth, with the motions which they are observed to have. On such foundations all the laws of natural philosophy rest.

12. Various illustrations of the law of reflexion have been given, more or less accurate; perhaps one of the highest confirmations of it is derived from the accordance of *Transit observations* of Stars made by reflexion at the surface of mercury with results obtained independently of reflexion. Of the law of refraction we shall speak hereafter when we come to explain the mode of determining the refractive indices of different substances.

The law of reflexion seems to have been known in very early times as we find it laid down in the earliest writers on Optics; the law of refraction was first accurately ascertained by Snell, about A. D. 1621, though it was first published by Descartes.

13. *Direct Reflexion and Refraction.*

Def. Let Q be the origin of a pencil of rays whose axis QA is incident directly (Art. 6) at A on a plane or spherical reflecting or refracting surface.



Then QA is the axis of the reflected or refracted pencil. Let QR be any ray incident at R whose direction after reflexion or refraction (produced if necessary) cuts the axis in q; since the normal at R lies in the plane QAR. Then as R is taken nearer and nearer to A, the point q will approach some point F in the axis, as its limiting position; and by taking R sufficiently near to A the distance Fq may be made less than any assignable magnitude. This limiting position (F) of the point q is called the *Geometrical Focus* of the reflected or refracted pencil.

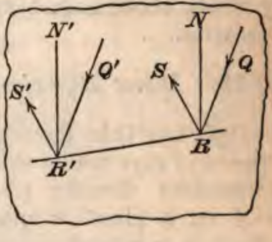
The point F will in some cases be nearer the surface than q is; in others more remote.

Def. The *principal focus* of a spherical reflecting or refracting surface is the geometrical focus of a pencil of parallel rays incident directly upon the surface parallel to a fixed diameter, called the *axis* of the surface.

Def. The *focal length* of a spherical reflecting or refracting surface is the distance between the surface and the principal focus,—measured along the axis.

The distance Fq is called the *aberration* of the ray Rq. If Ft be drawn perpendicular to the axis QA to meet Rq in t, it is sometimes convenient to distinguish Fq as the *longitudinal aberration* and Ft as the *lateral aberration.*

14. *A pencil of parallel rays will consist of parallel rays after reflexion at a plane surface.*



Let QR, $Q'R'$ be any two rays of a pencil of parallel rays incident on a plane reflecting surface at the points R, R'. Let RS be the direction of the ray QR after reflexion. Draw RN, $R'N'$ perpendicular to the surface at the points R, R', and (i) if the planes QRN, $Q'R'N'$ are *coincident*, draw $R'S'$ in this plane parallel to RS;

$$\text{then since } \left.\begin{matrix} R'N' \\ R'S' \end{matrix}\right\} \text{ are parallel to } \left\{\begin{matrix} RN \\ RS \end{matrix}\right. ;$$

$$\therefore \angle N'R'S' = \angle NRS = \therefore \angle QRN = \angle Q'R'N',$$

and therefore $R'S'$ is the direction of $Q'R'$ after reflexion; but (ii) if the planes QRN, $Q'R'N'$ do not coincide, let $R'S'$ be the intersection of the plane SRR' with the plane $Q'R'N'$.

Now $\left.\begin{matrix}QR\\RN\end{matrix}\right\}$ are parallel to $\left\{\begin{matrix}Q'R'\\R'N'\end{matrix}\right.$;

∴ the plane QRS is parallel to the plane $Q'R'S'$, (Euc. XI. 15),

...... st. line RS .... st. line $R'S'$, (Euc. XI. 16).

Also because $\left.\begin{matrix}QR\\RN\end{matrix}\right\}$ are parallel to $\left\{\begin{matrix}Q'R'\\R'N'\end{matrix}\right.$;

$$\therefore \angle QRN = \angle Q'R'N', \text{ (Euc. XI. 10).}$$

Similarly, $\angle SRN = \angle S'R'N'$.

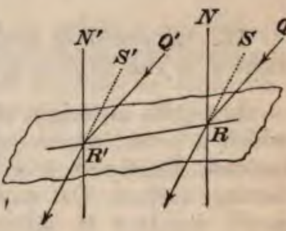
But $\angle QRN = \angle SRN$, (Art. 8),

$$\therefore \angle Q'R'N' = \angle S'R'N',$$

and $R'S'$ lies in the plane $Q'R'N'$; therefore it is the direction of the ray $Q'R'$ after reflexion; and it has been proved to be parallel to RS. But QR, $Q'R'$ are by supposition *any* two rays of the incident pencil; therefore the reflected pencil consists of parallel rays.

15. *A pencil of parallel rays consists of parallel rays after refraction at a plane surface.*

Let QR, $Q'R'$ be any two rays of a pencil of parallel rays incident on a plane refracting surface at the points R, R'. Let SR be the direction of the ray QR after refraction. Draw RN, $R'N'$ perpendicular to the surface at the points R, R', and let $S'R'$ be the intersection of the plane SRR' with the plane $Q'R'N'$ (if these planes be not coincident).



Now $\left.\begin{matrix}QR\\RN\end{matrix}\right\}$ are parallel to $\left\{\begin{matrix}Q'R'\\R'N'\end{matrix}\right.$;

∴ the plane QRS is parallel to the plane $Q'R'S'$;

∴ the straight line SR is parallel to the straight line $S'R'$.

Also because $\left.\begin{matrix}QR\\RN\end{matrix}\right\}$ are parallel to $\left\{\begin{matrix}Q'R'\\R'N'\end{matrix}\right.$;

$$\therefore \angle QRN = \angle Q'R'N'.$$

Similarly, $$\angle SRN = \angle S'R'N';$$

$$\begin{aligned}\therefore \sin \angle Q'R'N' &= \sin QRN\\ &= \mu \sin SRN\\ &= \mu \sin S'R'N'.\end{aligned}$$

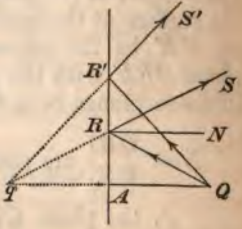
And $S'R'$ lies in the plane $Q'R'N'$, therefore $S'R'$ is the direction of the ray $Q'R'$ after refraction, and it has been proved parallel to SR.

If the planes SRR', $Q'R'N'$ be coincident, and $S'R'$ be drawn in this plane parallel to SR, it may readily be shewn that $S'R'$ is the direction of $Q'R'$ after refraction.

Now QR, $Q'R'$ are *any* two rays of the incident pencil, therefore the refracted pencil consists of parallel rays.

Remark. In the above figure the *dotted* lines SR, $S'R'$ are the directions of the refracted rays *produced backward:* it may not be amiss to suggest to the student that he will often avoid confusion in optical diagrams if he indicates by *dotted lines* any part of the directions of rays which are not *actually* traversed by the rays.

16. *A pencil is incident directly upon a plane reflecting surface, to find its form after reflexion.*



Let Q be an origin of light from which a pencil whose axis is QA is incident directly at A on a plane reflecting surface, QR any ray of the pencil incident at R, and reflected in direction RS in a plane passing through QR, and RN the normal to the surface at R. Since by the law of reflexion SR, QA lie in one plane, let them be produced to meet in q. Then since RN, qQ are parallel

$$\begin{aligned}\angle RQA &= \angle QRN\\ &= \angle SRN \text{ (Art. 8)}\\ &= \angle RqA;\end{aligned}$$

herefore the angles $\left.\begin{matrix}RqA\\RAq\end{matrix}\right\}$ are equal to $\left\{\begin{matrix}RQA\\RAQ\end{matrix}\right.$ each to each, nd RA is common to the two triangles;

$$\therefore Aq = AQ.$$

Now QR is *any* ray of the incident pencil; therefore the irections of all the rays of the reflected pencil (produced ackward) pass through the point q.

Hence the form of the reflected pencil is that of a cone hose axis is AQ, and vertex the point q—equidistant with Q rom the reflecting surface, and on the opposite side of it.

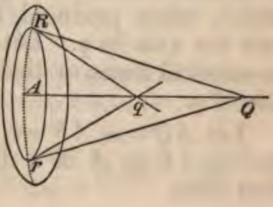
Cor. 1. Since the angles RqA, RQA are equal, the *ivergence* of the incident and reflected pencils is the same.

Cor. 2. If the incident pencil be *convergent*, a similar investigation will shew, that it will *converge* after reflexion to a point equidistant from the surface with the point of convergence of the incident pencil, and on the opposite side of it,—the degree of convergence being unaltered.

Cor. 3. The direction of the ray QR after reflexion cuts the axis in q, and the same is true in the limit when R moves up to A; therefore q is the geometrical focus of the reflected pencil.

17. The succeeding cases of direct reflexion and refraction have greater difficulty than the last investigation because in none of them will the pencil after reflexion or refraction pass accurately through a point. Our attention will at present be confined to plane and spherical surfaces.

A pencil whose origin is Q and axis QA incident directly on a plane or spherical reflecting or refracting surface may be regarded as composed of a series of conical surfaces or shells of rays, as QRr, with a common vertex Q and common axis QA. The rays of this conical surface will all be reflected or refracted similarly with respect



to QA, and therefore their directions after reflexion or refraction will form another conical surface with vertex q and axis Aq. Thus the reflected or refracted pencil will consist of a series of conical surfaces with a common axis, but different vertices. The limiting position of the vertex q is the geometrical focus of the pencil. (Art. 13.)

18. The determination of the form of a direct pencil after reflexion or refraction will consist of two parts:

(i) The determination of the geometrical focus.

(ii) The determination of the vertex q of any one of the cones of rays above described.

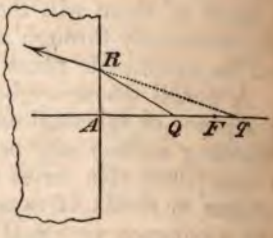
We will confine ourselves in the present chapter to the first of these parts—reserving the second for a subsequent chapter.

Obs. As it is obvious that a ray of light would traverse the same path *backward* if its course were *reversed* in any medium, it will be easily seen that Q (fig. Art. 13) would be the *geometrical focus* of a pencil of rays emanating from F the geometrical focus of Q. In this point of view, *the origin of a pencil* and *its geometrical focus* after reflexion or refraction are *convertible*, and we shall, for brevity, sometimes speak of them as *conjugate foci.*

19. *To find the geometrical focus of a pencil after direct refraction at a plane surface.*

Let Q be the origin of a pencil whose axis QA is incident directly at A on a plane refracting surface, then QA is the axis of the refracted pencil.

Let QR be any ray incident at R and refracted in a direction which, when produced backward, cuts the axis in q. Let F be the geometrical focus, or limiting position of q.



Let $AQ = u$, $AF = v$, lines being considered positive when measured from A in a direction contrary to that of the incident light.

Now RQA, RqA being equal to the angles of incidence and refraction of the ray QR,

$$\sin RQA = \mu \,.\, \sin RqA;$$

$$\therefore\ \frac{AR}{RQ} = \mu \,.\, \frac{AR}{Rq};$$

$$\therefore\ Rq = \mu \,.\, RQ.$$

In the limit when R moves up to A, and q to F,

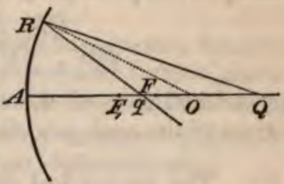
$$AF = \mu \,.\, AQ, \quad \text{or} \quad v = \mu u,$$

which formula gives the position of the geometrical focus of the refracted pencil.

20. *Obs.* In this proposition the pencil is considered *divergent*, as will be done in other cases, but attention to the algebraic sign of u will make the result applicable to a *convergent* pencil. In such a case u is negative and $v = \mu u =$ a *negative quantity*, indicates that the geometrical focus lies in a negative direction, i.e. behind the surface, at a distance from it determined by the numerical value of μu. The student will have little difficulty in drawing a suitable diagram and investigating the case independently.

21. *To find the geometrical focus of a pencil of rays after direct reflexion at a spherical surface.*

Let Q be the origin of a pencil of light whose axis QA is incident directly on a spherical reflecting surface of which O is the centre: then AQ is the axis of the reflected pencil. (Art. 17.)



Let QR be any ray incident at R, and reflected in direction Rq, cutting the axis in q; F the geometrical focus. Join OR.

Let $AQ = u$, $AO = r$, $AF = v$, lines being considered positive when measured from A in a direction contrary to that of the incident pencil.

Now $\angle\, QRO = \angle\, qRO$; $\therefore\ \dfrac{QR}{Rq} = \dfrac{QO}{Oq}$, (Euc. VI. 3), or ultimately when R moves up to A, and q to F,

$$AQ \,.\, OF = AF \,.\, OQ,$$

or
$$u(r-v) = v(u-r);$$

$$\therefore \frac{1}{v} - \frac{1}{r} = \frac{1}{r} - \frac{1}{u};$$

$$\therefore \frac{1}{v} + \frac{1}{u} = \frac{2}{r},$$

a formula which determines the position of F.

Cor. 1. If u be indefinitely great, *or* the incident pencil consist of parallel rays, the formula becomes

$$\frac{1}{v} = \frac{2}{r}, \text{ or } v = \frac{r}{2},$$

which assigns the position of the *principal focus* of the reflector, viz. at a point on its axis equidistant from the *centre* and the *surface*.

If F_1 be the principal focus $AF_1 = F_1O = \frac{r}{2}$.

Cor. 2. Since $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$, we get by reduction

$$uv - (u+v)\frac{r}{2} = 0; \text{ and } \therefore \left(u - \frac{r}{2}\right)\left(v - \frac{r}{2}\right) = \left(\frac{r}{2}\right)^2.$$

If F_1 be the *principal focus* this resultant may be written

$$QF_1 \,.\, FF_1 = AF_1^2,$$

or
$$FF_1 : AF_1 :: AF_1 : QF_1,$$

a geometrical form of the relation which exists between the positions of the *conjugate foci* Q, F.

22. The geometrical result of Art. 21, Cor. 2, can easily be obtained independently; thus, suppose a ray $Q'R$ incident *parallel to the axis* QA to be reflected in direction Rq' (these lines $Q'R$, Rq' are not drawn in the figure), then in the two triangles $Qq'R$, $Rq'q$, we have

$$\angle\, q'Rq = \angle\, QRQ' = \angle\, RQq',$$

$\angle Rq'q$ is common to the two triangles, which are there-similar, and we have

$$qq' : q'R :: q'R : q'Q;$$

therefore ultimately when R moves up to A, and q, q' to F_1,

$$FF_1 : AF_1 :: AF_1 : QF_1.$$

Obs. Since u, v are symmetrically involved in the equa- $\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$, if F were the origin of a pencil, Q would be geometrical focus after reflexion at the spherical mirror. points Q and F are thus convertible, and for this reason sometimes called *conjugate foci;* as was remarked before, . 18, *Obs.*

23. The case of a divergent pencil and concave mirror by ch the propositions of Art. 21 have been investigated, is sen because in it all the lines are measured in a positive ction. It will be seen that attention to the *signs* of u and r make this case include every other. We suppose in these es light to proceed from *right* to *left,* and *positive* lines con-iently to be measured from *left* to *right;* hence

(i) If the surface be *convex,* or O to the left of A, r is ative.

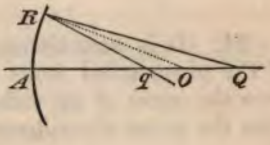
(ii) If the pencil be *convergent* or Q to the left of A, u is ative.

The *sign* which v has in the result in any case, determines which side of the point A the geometrical focus of the pen-ies, and the *magnitude* of v gives its distance from A.

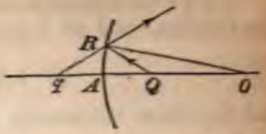
The four cases of the proposition are here subjoined as cises for the student to investigate independently.

i) Divergent pencil, Concave mirror, $\bigg\}$

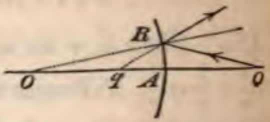
$\therefore \begin{cases} u \text{ is positive,} \\ r \text{ is positive.} \end{cases}$



F lies to the *right* or *left* of *A* according as $u \gtrless \frac{r}{2}$.

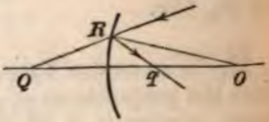


(ii) Divergent pencil, Convex mirror, $\therefore \begin{cases} u \text{ is positive,} \\ r \text{ is negative.} \end{cases}$



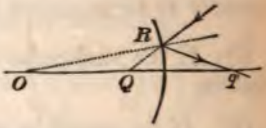
F always lies to the left of *A*.

(iii) Convergent pencil, Concave mirror, $\therefore \begin{cases} u \text{ is negative,} \\ r \text{ is positive.} \end{cases}$

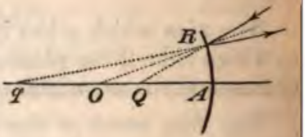


F always lies to the right of *A*.

(iv) Convergent pencil, Convex mirror, $\therefore \begin{cases} u \text{ is negative,} \\ r \text{ is negative.} \end{cases}$



F lies to the right or left of *A* as u is $\lessgtr \frac{r}{2}$.



Obs. All these results will follow from a discussion of the equation $\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$.

24. It is sometimes convenient to determine the position of the geometrical focus of a reflected pencil by its *distance from the centre* of the reflecting surface, instead of its distance the point of incidence.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] ptic, image of a straight line [illegible]

—*linear, angular* [illegible]

ge relatively to the object [illegible]

in the case of a spherical mirror [illegible]

, 152; length, &c. of a caustic, [illegible]

CHAPTER IX

On the Chromatic Dispersion of Light 142

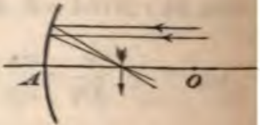
iment, 156; *pure spectrum,* how observed by Wollaston and [illegible], 157, 158; *fixed lines* of the spectrum, 159—161; mode of g the *refracting angle* of a prism, the *minimum deviation* of our—and the *index of refraction,* *test of the law of refraction,* 165; *standard scale* of colours, 166; Def. of *deviation,* *dis ationality of dispersion, secondary spectra,* 167; dispersion of s by a prism, 168; dispersive power, 171; *achromatic s,—possible* but *imperfect,* 172, 173; conditions of achr [illegible] *prisms,* 174—176; for *lenses,* 177—181; *circle of chromatic* 182—184; colours of natural bodies, primary colours, [illegible]

25. *To trace the relative change of position of the conju*
foci of a reflected pencil.

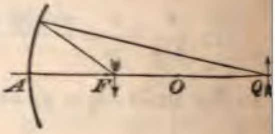
Suppose the reflecting surface concave, then the form
connecting the positions of the conjugate foci is $\frac{1}{v}+\frac{1}{u}=$
where the lines u, v, r are taken positive *to the right.*

It is clear from the formula that if u increases, v must
crease and *vice versâ*, i.e. *Q, F always move in opposite*
rections.

(i) Suppose $u=\infty$, i.e. the incident rays parallel, then $v=\frac{r}{2}$ and F coincides with the middle point of AO, the principal focus.

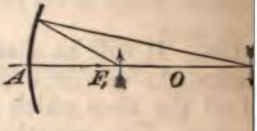


(ii) As Q moves from an indefinite distance up to O, F also moves up to O, and Q, F would coincide at O.

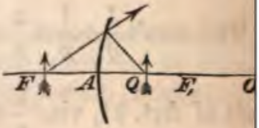


$\because$ if $u=r$ we must also have $v=r$.

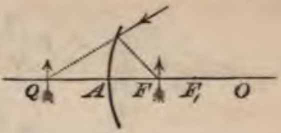
(iii) As Q moves from O to F_1, the middle point of AO, F moves off from O towards ∞, and v becomes ∞ when $u=\frac{r}{2}$, or Q coincides with F_1.



(iv) As Q continues to move in the same direction, from F_1 to A, v becomes negative and approaches from $-\infty$ towards A [illegible] F coincide at A.



(v) When u becomes negative and Q moves from A to ∞, v becomes positive again and moves from A towards F_1, and F incides with F_1 when u becomes ∞.

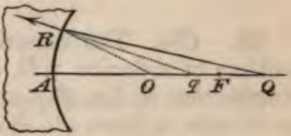


Similarly the change of relative sition may be traced when the flecting surface is *convex*.

26. *Obs.* In the diagrams of the preceding Art. a small bject has been placed in the position of Q, and its *geometrical age* in the position of F, as a guide to the student in tracing e form of such an image relatively to the object: underanding by *geometrical image*, the locus of the geometrical ci of pencils diverging from consecutive points of the object, d reflected at the surface.

This *geometrical image* will rarely coincide with the *visible age* as seen by an eye in a given position. In a subsequent apter we propose to give a method of determining the sible image in ordinary cases.

27. *To find the geometrical focus of a pencil of rays after rect refraction at a spherical surface.*



Let Q be the origin of a pencil light whose axis QA is incident irectly on a spherical refracting urface of which O is the centre: hen QA is the axis of the reacted pencil (Art. 17). Let QR e any ray incident at R and refracted in a direction which its QA in q, F the geometrical focus. Join OR. Let $AQ = u$, $O = r$, $AF = v$, lines being considered positive when measured om A in a direction contrary to that of the incident pencil.

$$\text{Now } \mu = \frac{\sin QRO}{\sin qRO} = \frac{\sin QRO \, . \, \sin ROq}{\sin ROQ \, . \, \sin qRO} = \frac{QO}{RQ} \cdot \frac{Rq}{Oq}.$$

$$\text{In the limit } \mu = \frac{QO \, . \, AF}{AQ \, . \, OF}.$$

$$\text{or } \mu u (v-r) = (u-r) v;$$

$$\therefore \mu \left(\frac{1}{r} - \frac{1}{v}\right) = \frac{1}{r} - \frac{1}{u},$$

$$\text{or } \frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r},$$

a formula which determines the position of F.

COR. 1. The position of the *principal focus* is obtained by making $u = \infty$, and we get

$$v = \frac{\mu}{\mu - 1} \cdot r.$$

COR. 2. To compare the degree of divergence of the pencil *after* and *before* refraction we must take the limit of the ratio $\angle RqA : \angle RQA$ when R approaches A;

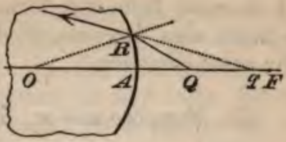
this ratio $= \dfrac{\sin RqA}{\sin RQA}$ ultimately $= \dfrac{RQ}{Rq}$ ultimately $= \dfrac{AQ}{AF} = \dfrac{u}{v}$.

If this result is negative it shews that the incident and refracted pencils are one *divergent* and the other *convergent.*

28. *Obs.* The remarks in Art. 23 in the corresponding problem of reflexion will, *mutatis mutandis*, apply in the case of refraction of Art. 27; and the student is recommended as an exercise to draw the diagrams and go through the investigation for each case that can occur,—with the surface *concave* or *convex* and the incident pencil *divergent* or *convergent.*

29. The following mode of expressing the position of the geometrical focus with respect to the *centre* of the refracting surface is sometimes convenient.

To find the distance of the geometrical focus of a pencil of rays from the centre after direct refraction at a spherical s



Let Q be the origin of a pencil of light whose axis QA is incident directly at A on a spherical refracting surface whose centre is O: then QA is the axis of the refracted pencil. Let QR be any ray, incident at R and refracted in a direction which cuts AQ in q, F the geometrical focus.

Let $OQ=p$, $OA=r$, $OF=q$, lines being considered positive when measured from O in a direction contrary to that of the incident pencil.

$$\text{Then } \mu=\frac{\sin QRO}{\sin qRO}=\frac{\sin QRO}{\sin ROQ}\cdot\frac{\sin ROq}{\sin qRO}=\frac{OQ\,.\,Rq}{RQ\,.\,Oq}.$$

$$\text{In the limit } \mu=\frac{OQ\,.\,AF}{AQ\,.\,OF},$$

$$\text{or, } \mu\,.\,AQ\,.\,OF=OQ\,.\,AF;$$

$$\text{i.e. } \mu q\,(p-r)=p\,(q-r),$$

$$\mu\left(\frac{1}{r}-\frac{1}{p}\right)=\frac{1}{r}-\frac{1}{q},$$

$$\therefore\ \frac{1}{q}-\frac{\mu}{p}=-\frac{\mu-1}{r},$$

a formula which determines the position of F with respect to O.

Obs. The result of this article is very convenient in the case of refraction through a sphere (see Art. 105).

30. *To trace the relative change of position of the conjugate foci of a refracted pencil.*

Suppose the refracting surface concave, then the formula connecting the positions of the conjugate foci is

$$\frac{\mu}{v}-\frac{1}{u}=\frac{\mu-1}{r},$$

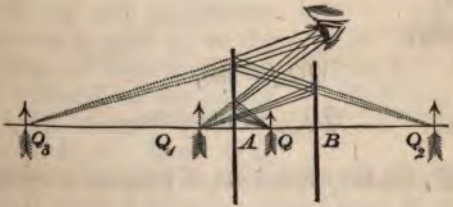
The *remarks* of Art. 26 apply also to this case.

31. The reflecting or refracting surfaces have been considered spherical in the preceding articles, because such are the surfaces which generally occur in the construction of optical instruments. When the surface is any other figure of revolution, it is in general sufficient, when the pencil is not very large, to consider the surface the same as the spherical surface generated by the circle of curvature at the vertex of the generating curve.

The case of a reflecting paraboloid of revolution is however of some interest, inasmuch as its properties are sometimes employed in large telescopes, like that of Lord Rosse.

Illustrations and Examples.

32. *A luminous point being placed between two parallel plane mirrors, to find the position of the images formed by successive reflexions at the mirrors.*



Through the luminous point Q let the line AQB be drawn perpendicular to the two mirrors A, B and produced indefinitely both ways.

Then taking $AQ_1 = AQ$, Q_1 will be the geometrical focus or image of rays emanating from Q and reflected at the mirror A. The rays so reflected from A and diverging from Q_1 will be incident on the second mirror B, and if we take $BQ_2 = BQ_1$, Q_2 will be the focus of the rays after reflexion from B. The rays will then be incident upon A, diverging apparently from Q_2 and be reflected at A, from a point Q_3 such that $AQ_3 = AQ_2$, and so on.

Again, the rays diverging from Q and incident on the second mirror B will have a geometrical focus or image in Q', where $BQ = BQ'$; the rays diverging from this point Q' and incident on the first mirror A, will after reflexion at A diverge from a point Q'' such that $AQ'' = AQ'$, and so on. Thus there are two sets of images, each infinite in number, all arranged on the line AB and becoming more and more distant after each reflexion. (The second set are not put in the figure, to avoid confusion.)

The distances QQ_1, QQ_2 ... may be easily calculated. For putting $QA = a$, $QB = b$, $AB = a + b = c$, we have

$$QQ_1 = 2AQ = 2a,$$
$$QQ_2 = BQ + BQ_1 = QQ_1 + 2BQ = 2a + 2b = 2c,$$
$$QQ_3 = AQ + AQ_2 = QQ_2 + 2AQ = 2c + 2a,$$
$$QQ_4 = \dots = 4c.$$

And generally,

$$QQ_{2n} = 2nc,$$
$$QQ_{2n+1} = 2nc + 2a.$$

Similarly, for the second set of images, we should have

$$QQ' = 2b,\ QQ'' = 2c,\ QQ''' = 2c + 2b,\ QQ^{\text{iv}} = 4c,$$
$$QQ^{(2n)} = 2nc,\ QQ^{(2n+1)} = 2nc + 2b.$$

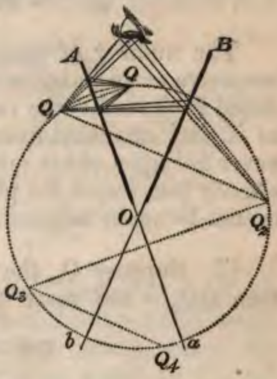
If Q is midway between A and B, $a = b$, and the two sets will form a series which are successively at equal distances ($= c$) from each other.

Obs. The diagram will be sufficient to indicate the manner in which the images are seen by an eye in a given position, the small pencil received by the eye passing between the reflexions in the same manner as if it diverged from the successive images.

A familiar illustration of the above may be noticed in a drawing-room where there are two mirrors on opposite walls, parallel to each other,—an interminable series of images of the objects in the room is observed.

33. *A luminous point being placed between two plane mirrors inclined to one another at a given angle, to find the position and number of the images formed by successive reflexions at the mirrors.*

Let the figure represent a section of the mirrors A, B by a plane passing through the luminous point Q, and perpendicular to each mirror, and therefore perpendicular to the line of intersection—which is represented by the point O.



Describe a circle with centre O and passing through Q. Draw QQ_1 perpendicular to the first mirror A, and meeting the circle in Q_1. Then Q, Q_1 are equidistant from the mirror A, and Q_1 is the *geometrical focus* of rays diverging from Q and reflected at the mirror A, i.e. Q_1 is the image of Q formed by reflexion at A. Draw Q_1Q_2 perpendicular to the second mirror B, then in like manner Q_2 is the image formed by rays reflected from B, after reflexion at A; the course of the rays between the two reflexions at A, B being the same as if they diverged from Q_1. Similarly drawing Q_2Q_3 perpendicular to A or A produced, we obtain a third image Q_3 formed by rays reflected at A, B, and again at A, and so on. Thus we get a set of images formed by rays *first* reflected at A, then at B, again at A, and so on in succession.

In like manner we should get a second set of images formed by rays *first* reflected at B, then at A, again at B, and so on in succession. To avoid confusion we have not marked this second set of images in the figure.

[illegible]

[illegible]

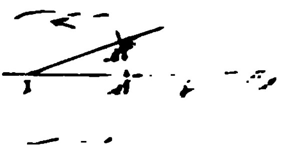
$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]



Let Q be the origin of a pencil [illegible] whose axis QA is incident [illegible] at A on a spherical refracting surface whose centre is [illegible] then QA is the axis of the [illegible] pencil. Let QR be any [illegible] incident at R and refracted [illegible] direction which meets AO in q, [illegible] the geometrical [illegible]

Let $OQ=p$, $Oq=$ [illegible] $=r$, lines [illegible] positive when measured from O in a direction [illegible] the incident pencil.

Then $\mu = \frac{\sin QRO}{\sin qRO} =$ [illegible]

In the limit $\mu = \frac{OQ \cdot Aq}{AQ \cdot Oq}$

$$\text{or, } \mu \cdot AQ \cdot Oq = OQ \cdot Aq$$

[illegible]

[illegible]

[illegible]

a formula which determines the position of q [illegible] to O.

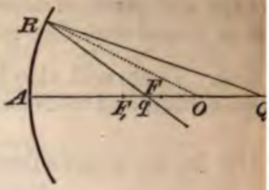
Obs. The results [illegible] case of refraction through a [illegible]

30. *To trace the relative course of* [illegible] *foci of a refracted pencil.*

Suppose the refracting surface [illegible] connecting the positions of the conjugate [illegible]

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r};$$

Since AR $(=y)$ ultimately vanishes, it is *very small* co pared with the other lines in the figure, u, v, r. Hence we may (neglecting cubes of small quantities and therefore putting $\sin\theta=\theta$) write $\angle RqA=\frac{y}{v}$, $\angle ROA=\frac{y}{r}$, $\angle RQA=\frac{y}{u}$,



and since $\angle QRO=\angle ORq$,

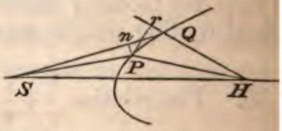
$$\therefore \angle RqA-\angle ROq=\angle qRO=\angle ORQ$$
$$=\angle ROA-\angle RQA;$$
$$\therefore \frac{y}{v}-\frac{y}{r}=\frac{y}{r}-\frac{y}{u}, \text{ or } \frac{1}{v}-\frac{1}{r}=\frac{1}{r}-\frac{1}{u},$$
$$\text{or } \frac{1}{v}+\frac{1}{u}=\frac{2}{r}.$$

This equation which is approximately true when y small, becomes strictly true in the limit when $y=0$.

In a similar manner it may be shewn that the relat $\frac{\mu}{v}-\frac{1}{u}=\frac{\mu-1}{r}$ (Art. 27), when put in the form $\frac{y}{r}-\frac{y}{u}=\mu\left(\frac{y}{r}-\right.$ expresses simply the law of refraction, $\sin\phi=\mu\sin\phi'$, $\phi=\mu\phi'$, when the angles ϕ, ϕ' are taken indefinitely smal

36. *Rays diverge from a point S; to find a surface wh will refract them accurately to another given point H.*

The surface will be one of revolution about the line joi the two points S, H of divergence and convergence, as axi



Let SP, SQ be two contiguous rays diverging from S and refracted at P, Q accurately to the point H, in one plane passing through S, H,—μ the refractive index from the first medium into the second. Draw Pn perpendicular to SQ, and Pr perpendicular to HQ produced.

n we may regard Qn as the *increment* of SP, and Qr lecrement of HP, in passing from P to Q; one of the es SP, PH being increased and the other diminished, ie normal at P must fall between S and H. ϕ, ϕ' the of incidence and refraction at P, and PQ so small as to rded as coincident with its chord.

n
$$\delta . SP = \quad Qn = \quad PQ . \sin \phi,$$
$$\delta . HP = - Qr = - PQ . \sin \phi';$$
$$\delta . SP + \mu . \delta . HP = PQ (\sin \phi - \mu \sin \phi') = 0,$$

imit when Q moves up to P.

ce, integrating

$$SP + \mu . HP = C \text{ a constant} \ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

s equation is that of a curve which by revolution about ierated a surface such as is required; and which is ily called an *aplanatic* surface.

. 1. If the rays diverging from S, are to *diverge* point H after refraction, the curve required would be

$$SP - \mu . HP \; C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

. 2. If the surface is required which will *reflect* rays ng from S, accurately *to* or *from* H, we should obtain forms of the generating curves $SP + HP = C$, and $P = C$ respectively, by proceeding in a manner similar bove.

se results can of course be obtained independently by ivestigation. They represent respectively an ellipse yperbola; and it may be easily shewn *inversely* that erging from one focus of an ellipse converge to the cus after reflexion at the curve, and that rays diverg- n one focus of a hyperbola will after reflexion at the verge from the other focus.

rticular case of either of these, is a paraboloid of re- in which rays incident on the concave side of the and parallel to the axis will be reflected to the focus, and *vice versâ*. This property of a paraboloid has beer ployed in forming specula for reflecting telescopes.

COR. 3. If the equation $SP-\mu . HP=C$ were expr in rectangular co-ordinates, it would give an algebraic c of the fourth degree. In the particular case of $C=0$, surface will be a sphere.

The discussion of *aplanatic* surfaces was originally purs by Newton and Descartes,—in fact the class of curves inclu in the equation (i) are frequently called *Cartesian ovals*; with the exception of paraboloids referred to in the previ corollary, they are now become questions of curiosity.

See Newton's *Principia*, Bk. I. § 14, Prop. 97.

CHAPTER II.

ILLUMINATION OF SURFACES.

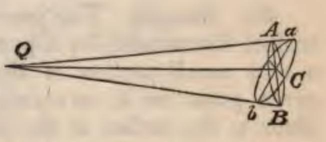
37. WHEN a pencil of light emanates from a luminous int and is propagated in a uniform medium if we suppose intensity unaltered by the absorption of any portion of by the medium, yet from other causes the illumination at y point of a surface exposed to the light is different in fferent positions and at different distances from the surface.

38. *Def.* The illumination at any point of a surface posed to light is measured by a quantity I: $I\kappa$ being the nount of illumination of an indefinitely small area κ of the rface contiguous to the point in question,—some standard gree of illumination being referred to as a unit.

Hence, if the same quantity of light fall on two very nall areas,—$I\kappa$ being the same for each,—the illuminations any point of these areas are inversely as the areas.

39. *When a small plane area is illuminated by a pencil rays emanating from a point, the illumination at any point the area*

$$\propto \frac{\textit{cosine of angle of incidence}}{(\textit{distance from origin})^2}.$$



Let QAB be a small conical encil of light from an origin Q, CB, aCb a circular and oblique ection of it through a point C n the axis of the pencil.

If aCb be a small plane area illuminated by the pencil,

(i) In all sections parallel to aCb, the quantity of light in the pencil being the same, the illumination at a point is inversely as the area of the section (Art. 38); i. e. inversely as (dist.)^2 from Q.

(ii) In all sections through C at different inclinations to the axis, the quantity of light received being the same as that received by ACB, the illumination varies as the area inversely. But if the pencil be supposed so small that ACB may be regarded as the orthogonal projection of aCb,

$$\text{area } aCb = \frac{\text{area } ACB}{\cos \angle ACa},$$

$$\therefore \text{ illumination in } aCb = \text{illumination in } ACB \,.\, \cos \angle ACa$$
$$\propto \cos \angle ACa,$$

and $\angle ACa$ being the inclination of the planes ACB, aCb, is the angle between the perpendiculars to these planes at C, *or* is the angle of incidence of QC.

Hence, when both the distance and angle of incidence vary together,—the illumination at any point of the area

$$\propto \frac{\text{cosine angle of incidence}}{(\text{distance})^2}.$$

COR. The illumination at any point of the area

$$= C \,.\, \frac{\text{cosine } \angle \text{ of incidence}}{(\text{distance})^2},$$

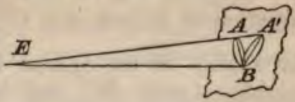
where C depends only on the brightness of the illuminating point, and is the illumination at any point of a small area directly exposed to the pencil at a distance *unity* from the origin.

40. *Remark.* The preceding result will apply to a curved surface illuminated by a pencil of light, since we may regard a *very small* portion of the curved surface contiguous to a given point on it, as coincident appreciably with the tangent plane to the surface at the point considered.

41. It is found by experiment that luminous surfaces appear to be of the same brightness at any point, whatever be the inclination of the surface at that point to the axis of the pencil by which it is seen. The Sun's disc, *for example*, is equally bright at all distances from the centre. The apparent

trinsic brightness of a bar of red-hot iron is not sensibly tered by inclining it obliquely to the eye.

Thus if $A'B$ be a *small* portion of a luminous surface iewed *obliquely* by an eye E, the mount of light received from it ill be the same as that received om a portion AB viewed *di-ectly:* AB being the section, erpendicular to the axis of the cone, of which E is the ertex and $A'B$ the base; i. e. the intensity of emission of ight in direction $A'E$: intensity of emission perpendicular to he surface at A' :: area AB : area $A'B = \sin\theta : 1$, where θ s the angle which EA' makes with the luminous surface at A'.

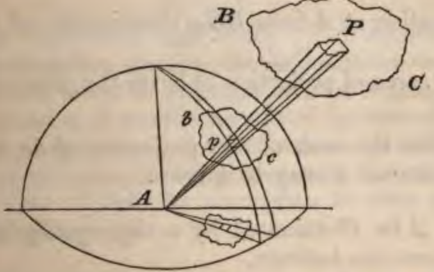


Hence it follows that *the copiousness of emission of light rom a luminous surface is proportional to the sine of the angle f emanation from the surface.*

42. The following is a mode of calculating the illumination at any point of a surface illuminated by a given surface of uniform brightness.

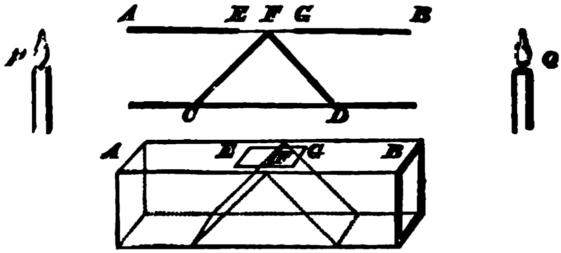
Let A be a small plane area illuminated by a surface BC of uniform brightness.

About A as centre describe a sphere, and let a line through A moving round the boundary of BC intersect the surface of this sphere in the curve bc. Take also an element



P of the surface BC, and let p be the corresponding element f the spherical surface formed as before. Let α, θ be the

This instrument consists of a rectangular box, about tw inches square, open at both ends, of which *ABCD* is a s



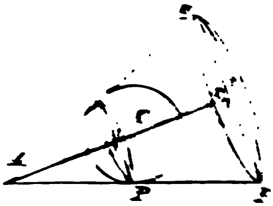
tion. The inner surface is blackened so as to absorb extra-neous light. Within the box inclined at angles of 45° to i sides are placed two rectangular pieces of looking-glass, *FC*, *FD*, *cut from one and the same rectangular strip* to ensure the exact equality of their reflecting powers, and fastened so a to meet at *F*, in the middle of a narrow slit *EFG* about a inch long and one-eighth of an inch broad, which is covered with a slip of fine tissue or oiled paper. The rectangular slit should have a slip of blackened card at *F*, to prevent the lights reflected from the looking-glasses mingling with each other.

When we wish to compare the illuminating powers of two sources of light (two flames for instance)—*P* and *Q*, they must be placed at such a distance from each other, and from the instrument between them, that the light from every part of each shall fall on the reflector next it; and be reflected to the corresponding portion of the paper *EF* or *FG*. The instrument is then to be moved nearer to one or the other, till the paper on either side of the division *F* appears equally illuminated. To judge of this it should be viewed through a tube blackened within, one end resting upon the upper part *EFG* of the photometer, the other applied quite close to the eye. When the lights are thus exactly equalized, it is clear that the total illuminating powers of the luminaries are directly as the squares of their distances from the middle of the instrument.
]

Note. To render the comparison of the lights more exact the equalization of the lights should be performed several times, turning the instrument end for end each time. The mean of the several determinations will then be very near the truth.

By means of the above instrument we may obtain an easy experimental proof of the decrease of light as the inverse square of the distance. For if we place four candles at *E* and one at *G* (as nearly equal as possible and burning with equal flame) it is found that the portions of the paper *EF*, *GF* will be equally illuminated when the distances *EF*, *GF* are as 2 : 1, and so for any number of candles at each side.

46. A plane surface touches a self-luminous sphere: to find the illumination of the surface at any point.



Let *A* be any point in the plane surface which the sphere whose centre is *C* touches at *P*. Join *CP*. With centre *A* describe a spherical surface, and let a straight line through *A* moving round the sphere *C* so as to define the portion of it from which *A* receives illumination, intersect the spherical surface described about *A* in the small circle *pq*, the plane of which meets *AC* or *AC* produced in *c*.

Then projection on plane *AP*, of the curved surface *pq*,

$$= \text{projection of the plane circle } pq$$

$$= \pi . cp^2 . \frac{cp}{Ap}. \quad \text{(Hymers. \textit{Three Dim.} Art. 81).}$$

Therefore illumination at *A* from the sphere (Art. 42)

$$= \frac{C}{Ap^2} . \frac{\pi . cp^2}{Ap} = \pi . C . \left(\frac{cp}{Ap}\right)^3 = \pi C . \left(\frac{CP}{AC}\right)^3 . \propto \frac{1}{AC^3}.$$

49. We premise the following theorem for the purpose of shewing that an approximate value of a quantity to be determined, may be substituted in the small terms of the equation.

Suppose v a quantity whose value is to be found, and which is given implicitly by an equation of the form

$$v = V + f(v) \cdot y^2 \text{.....................} (A),$$

where V involves known quantities only, and is independent of y, the small quantity by powers of which the approximation is conducted,—and $f(v)$ is a function of v and known quantities.

Then by substitution

$$f(v) = f\{V + f(v) \cdot y^2\} = f(V) + f'(V) \cdot f(v) \cdot y^2 + \dots$$

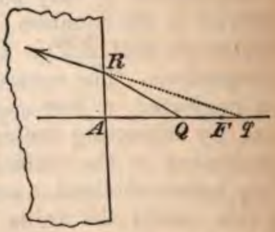
by Taylor's Theorem;

$$\therefore v = V + f(V) \cdot y^2 + f'(V) \cdot f(v) \cdot y^4 + \dots$$
$$= V + f(V) \cdot y^2,$$

if the approximation extend only to the square of y.

It appears therefore that in the term of (A) which involves y^2, we may substitute the value of v obtained by neglecting y^2, and the result will be true to the order of y^2.

50. *When a pencil is incident directly on a plane refracting surface, to find the point where the direction of a given ray after refraction cuts the axis.*



Let Q be the origin of a pencil whose axis QA is incident directly at A on a plane refracting surface, then QA is the axis of the refracted pencil. Let QR be any ray incident at R, and refracted in a direction which cuts the axis in q, the position of which is to be determined.

Let $AQ = u$, $Aq = v'$; lines being considered positive when from A in a direction contrary to that of the incident pencil. Also let $AR = y$, a quantity of which the cube and higher powers may be neglected.

Now RQA, RqA being equal to the angles of incidence and refraction of the ray QR,

$$\sin RQA = \mu . \sin RqA;$$

$$\therefore\ Rq = \mu . RQ,$$

$$\text{or } \sqrt{(v'^2 + y^2)} = \mu \sqrt{(u^2 + y^2)};$$

$$\therefore\ v'\left(1 + \frac{y^2}{2v'^2}\right) = \mu u \left(1 + \frac{y^2}{2u^2}\right), \text{ approximately,}$$

$$v' = \mu u + \left(\frac{\mu}{u} - \frac{1}{v'}\right)\frac{y^2}{2}.$$

In the coefficient of y^2, for v' we may substitute μu, the first approximate value of v', and the resulting equation will be true to the order of y^2, (Art. 49);

$$\therefore\ v' = \mu u + \left(\frac{\mu}{u} - \frac{1}{\mu u}\right)\frac{y^2}{2}$$

$$= \mu u + \frac{\mu^2 - 1}{\mu} . \frac{y^2}{2u}.$$

Cor. Hence it appears by comparing this result with that of Art. 19, that the geometrical focus is the approximate position of the point q, when powers of y above the *first* are neglected.

51. When a pencil is directly reflected or refracted at a surface, the aberration of any ray is the distance between the geometrical focus and the point where the direction of that ray after reflexion or refraction cuts the axis (13).

The aberration of a *pencil* is the aberration of the extreme ray in any section of the pencil through its axis.

ter reduction, since cos *ROA*

$$=\cos\frac{y}{r}=1-\frac{y^2}{2r^2},\ \text{approximately};$$

$$\therefore\ QR=u\left\{1-\left(\frac{1}{r}-\frac{1}{u}\right)\frac{y^2}{2u}\right\}.$$

Similarly, $$qR=v'\left\{1-\left(\frac{1}{r}-\frac{1}{v'}\right)\frac{y^2}{2v'}\right\};$$

$$(r-v')\,u\left\{1-\left(\frac{1}{r}-\frac{1}{u}\right)\frac{y^2}{2u}\right\}=(u-r)\,v'\left\{1-\left(\frac{1}{r}-\frac{1}{v'}\right)\frac{y^2}{2v'}\right\}.$$

If each side of this be divided by $u\,v'\,r$

$$\left(\frac{1}{v'}-\frac{1}{r}\right)\left\{1-\left(\frac{1}{r}-\frac{1}{u}\right)\frac{y^2}{2u}\right\}=\left(\frac{1}{r}-\frac{1}{u}\right)\left\{1-\left(\frac{1}{r}-\frac{1}{v'}\right)\frac{y^2}{2v'}\right\},$$

$$\frac{1}{v'}+\frac{1}{u}=\frac{2}{r}+\left(\frac{1}{r}-\frac{1}{u}\right)\left(\frac{1}{v'}-\frac{1}{r}\right)\left(\frac{1}{u}+\frac{1}{v'}\right)\frac{y^2}{2}.$$

If we neglect powers of y higher than the first

$$\frac{1}{v'}+\frac{1}{u}=\frac{2}{r},\quad \text{or}\quad v'=v\quad \text{(Art. 21)},$$

hich value we may substitute in the coefficient of y^2, and the esulting equation will be true as far as y^2;

$$\therefore\ \frac{1}{v'}+\frac{1}{u}=\frac{2}{r}+\left(\frac{1}{r}-\frac{1}{u}\right)^2.\frac{y^2}{r},$$

hich gives the position of q.

Cor. 1. $$\frac{1}{v'}-\frac{1}{v}=\left(\frac{1}{r}-\frac{1}{u}\right)^2.\frac{y^2}{r};$$

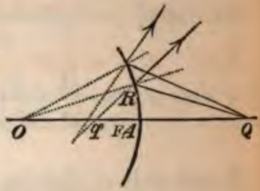
$$\therefore\ v'-v=-\left(\frac{1}{r}-\frac{1}{u}\right)^2.\frac{y^2}{r}.vv',$$

or putting $v' = v$ in the term involving y^2, we have

$$\textit{aberration of ray } RQ = v' - v = -\left(\frac{1}{r} - \frac{1}{u}\right)^2 \cdot \frac{v^2 y^2}{r}$$

$$= -\frac{\left(\frac{1}{r} - \frac{1}{u}\right)^2}{\left(\frac{2}{r} - \frac{1}{u}\right)^2} \cdot \frac{y^2}{r}.$$

Cor. 2. If, as in Art. 24, the position of any ray is determined with reference to the centre O of the surface, we should get



$$\frac{r}{Oq} + \frac{r}{p} = 2 \cos \theta.$$

Write $Oq = q'$, then

$$\frac{r}{q'} + \frac{r}{p} = 2 \cos \theta,$$

$$\text{and } \frac{r}{q} + \frac{r}{p} = 2 \text{ (Art. 24)};$$

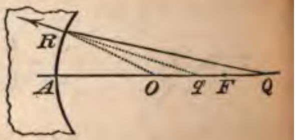
$$\therefore \frac{1}{q} - \frac{1}{q'} = \frac{2}{r}(1 - \cos \theta) = \frac{\theta^2}{r}, \text{ nearly,}$$

$$= \frac{y^2}{r^3}, \text{ nearly}; \quad \because \theta = \frac{y}{r};$$

$$\therefore \text{ aberration of ray } qR = q' - q = \frac{y^2 \cdot q^2}{r^3}.$$

53. *When a pencil is incident directly on a spherical refracting surface,—to find the point where the direction of a given ray after refraction cuts the axis.*

Let Q be the origin of a pencil whose axis QA is incident directly at A on a spherical refracting surface whose centre is O; then QA is the axis of the refracted pencil. Let QR be any ray incident at R, and refracted in a direction which cuts the axis in q, the position of which is to be determined. Join OR.



Let $AQ=u$, $Aq=v'$, $AO=r$, lines being considered ositive when measured from A in a direction opposite to hat of the incident pencil. Also let $AR=y$—a quantity of which the cube and higher powers may be neglected.

$$\text{Then } \mu=\frac{\sin QRO}{\sin qRO}=\frac{\sin QRO \,.\, \sin ROq}{\sin ROQ \,.\, \sin qRO}=\frac{OQ \,.\, Rq}{RQ \,.\, Oq},$$

r
$$\mu \,.\, RQ \,.\, Oq = OQ \,.\, Rq.$$

$$\text{Now}\quad RQ^2=RO^2+OQ^2+2RO\,.\,OQ\cos ROA$$

$$=r^2+(u-r)^2+2r(u-r)\cos\frac{y}{r}$$

$$=u^2-(u-r)\frac{y^2}{r},\text{ nearly;}$$

$$\therefore\ RQ=u\left\{1-\left(\frac{1}{r}-\frac{1}{u}\right)\frac{y^2}{2u}\right\}.$$

Similarly, $$Rq=v'\left\{1-\left(\frac{1}{r}-\frac{1}{v'}\right)\frac{y^2}{2v'}\right\};$$

$$\therefore\ \mu(v'-r)\,u\left\{1-\left(\frac{1}{r}-\frac{1}{u}\right)\frac{y^2}{2u}\right\}=(u-r)\,v'\left\{1-\left(\frac{1}{r}-\frac{1}{v'}\right)\frac{y^2}{2v'}\right\};$$

$$\therefore\ \mu\left(\frac{1}{r}-\frac{1}{v'}\right)\left\{1-\left(\frac{1}{r}-\frac{1}{u}\right)\frac{y^2}{2u}\right\}=\left(\frac{1}{r}-\frac{1}{u}\right)\left\{1-\left(\frac{1}{r}-\frac{1}{v'}\right)\frac{y^2}{2v'}\right\},$$

or
$$\frac{\mu}{v'}-\frac{1}{u}=\frac{\mu-1}{r}+\left(\frac{1}{r}-\frac{1}{v'}\right)\left(\frac{1}{r}-\frac{1}{u}\right)\left(\frac{1}{v'}-\frac{\mu}{u}\right)\frac{y^2}{2}.$$

If the square of y be neglected, v the approximate value of v' is given by the equation

$$\frac{\mu}{v}-\frac{1}{u}=\frac{\mu-1}{r},$$

which value of v' may be used in the coefficient of y^2 and the esult will be true as far as y^2.

$$\therefore \frac{\mu}{v'}-\frac{1}{u}=\frac{\mu-1}{r}+\frac{1}{\mu^2}\cdot\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{u}+\frac{\mu-1}{r}-\frac{\mu^2}{u}\right)\frac{y^2}{2}$$

$$=\frac{\mu-1}{r}+\frac{\mu-1}{\mu^2}\cdot\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)\frac{y^2}{2},$$

which equation determines the position of q.

COR. $\frac{\mu}{v'}-\frac{\mu}{v}=\frac{\mu-1}{\mu^2}\cdot\left(\frac{1}{r}-\frac{1}{u}\right)^2\cdot\left(\frac{1}{r}-\frac{\mu+1}{u}\right)\frac{y^2}{2}$;

therefore the aberration of the ray $Rq'=v'-v$

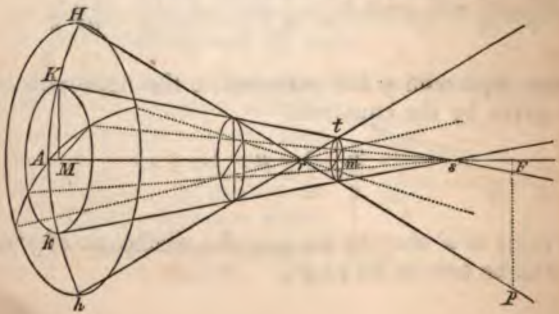
$$=-\frac{\mu-1}{\mu^3}\cdot\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)\frac{y^2v^2}{2}$$

$$=-\frac{\mu-1}{\mu}\,\frac{\left(\frac{1}{r}-\frac{1}{u}\right)^2}{\left(\frac{\mu-1}{r}+\frac{1}{u}\right)^2}\left(\frac{1}{r}-\frac{\mu+1}{u}\right)\frac{y^2}{2}.$$

54. *Obs.* In the preceding Articles, since powers above the second have been neglected, it is immaterial ther we take for y the length of the arc AR, or the per dicular distance of R from the axis of the surface.

Least Circle of Aberration.

55. Let AF be the axis of a pencil of rays reflecte refracted directly at a spherical surface; Hr, hr the extr rays in any plane section of the pencil through the



which meet in the point r of the axis; Ks, ks any other two ays in the same section meeting in the point s of the axis. Suppose hr (produced if necessary) to cut Ks in t and draw m perpendicular to AF.

Now (i) considering rays incident at different points along AH, when K coincides either with A or H, tm is $=0$; therefore for some position of K in AH, tm is a maximum.

(ii) If K have the position for which tm is a maximum, a circle with centre m and radius mt, in a plane perpendicular to AF, is the smallest space through which the whole pencil passes. For a circular section of the pencil *to the left* of m is larger in consequence of the converging cone Ksk, and a circular section *to the right* of m is larger in consequence of the diverging cone trm. Such a circle is called the *Least Circle of Aberration* of the directly reflected or refracted pencil.

Obs. In the preceding figure, the aberration is supposed to be *towards* the surface; the student will have little difficulty n drawing a corresponding figure for the case when the aberration is *from* the surface.

In the following calculation of the position and dimensions of this circle, the cube and higher powers of the half-breadth of the pencil will be neglected, as has been done in former cases.

56. *To calculate the position and dimensions of the Least Circle of Aberration after direct reflexion or refraction, at a plane or spherical surface.*

Let $AK=y'$, $AH=y$, $r=$ rad. of spherical surface AK (fig. Art. 55). Draw KM perpendicular to AF.

$$\therefore\ KM = r\sin\frac{y'}{r} = y',$$

$$AM = r\text{ vers.}\frac{y'}{r} = \frac{y'^2}{2r},\text{ nearly.}$$

By similar triangles

$$\frac{ms}{mt}=\frac{Ms}{MK}=\frac{As-\frac{y'^2}{2r}}{y'}=\frac{As}{y'}-\frac{y'}{2r};$$

$$\therefore\ ms=mt.\left(\frac{As}{y'}-\frac{y'}{2r}\right).$$

So $$\frac{mr}{mt}=\frac{Mr}{AH}=\frac{Ar}{y}-\frac{y}{2r};$$

$$\therefore\ mr=mt\left(\frac{Ar}{y}-\frac{y}{2r}\right).$$

Now since mt, depending on the aberration, is at least of the order y'^2,—therefore $y'.mt$ and $y.mt$ may be neglected. Also the difference between Ar and As being the difference of the aberrations of the two rays Hr, Ks, is of the order y'^2; and therefore when multiplied by mt we may regard Ar and As as equal, since the error introduced by so doing is of a higher order than y'^3.

Hence we may take

$$rs=rm+ms=mt.Ar\left(\frac{1}{y}+\frac{1}{y'}\right).$$

But (Art. 52, 53) the aberration of any ray in the same pencil $\propto y^2$;

$$\therefore\ Fs : Fr=y'^2 : y^2.$$

$$\therefore\ rs : Fr=y^2-y'^2 : y^2;$$

$$\therefore\ Fr.\frac{y^2-y'^2}{y^2}=rs=mt.Ar.\frac{y+y'}{yy'};$$

$$\therefore\ tm=\frac{Fr}{Ar}.\frac{(y-y')\,y'}{y}.$$

Now mt is variable by the change of y', and this value of mt will therefore be a maximum when

$$y-2y'=0,\ \text{i.e.}\ y'=\frac{y}{2};$$

$$\therefore\ mt = \frac{1}{4}\cdot\frac{Fr}{Ar}\,y \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)},$$

$$rm = \frac{1}{4}\cdot Fr \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)};$$

hich two equations define the position and magnitude of the *ast circle of aberration.*

In other words, the distance of the circle of least aberration rom the geometrical focus is three-fourths of the longitudinal berration of the pencil, and its radius is one-fourth of *Fr*, he lateral aberration of the extreme ray.

heir ultimate intersections may be obtained in the same way as a common envelope. This envelope is the *caustic curve*, by the revolution of which about the axis the *caustic surface* is generated.

Caustics formed by reflexion were formerly called *catacaustics;* those formed by refraction, *diacaustics.*

59. Since the surfaces with which we have to deal are practically limited in extent, it will be convenient to give a few definitions before we proceed to examine the form of oblique pencils, and we shall suppose the surfaces to be either *plane* or *spherical.*

Def. *Oblique* incidence on a reflecting or refracting spherical surface is either *centrical* or *excentrical.* A pencil is incident *centrically* when the axis of the pencil is incident at a definite point of the surface, called *the centre of the face;* in other cases it is incident *excentrically.*

A distinction must be preserved between the *centre of the surface* and the *centre of the face;* the former is the centre of the sphere of which the reflecting or refracting surface is a portion; the latter is a point on the surface itself, generally the point with respect to which the surface is symmetrical.

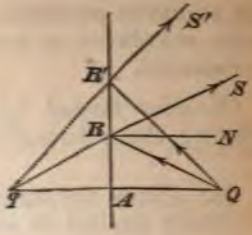
Def. The diameter of the spherical surface which passes through the centre of the face, is in general called the *axis* of the reflecting or refracting surface.

The circumstance which renders the calculation of centrical pencils less complex than those of excentrical pencils is, that in the former case the point of incidence is a definite point of reference from which lines may be conveniently measured, but in the latter it is not so.

60. *If a divergent pencil be incident obliquely on a plane reflecting surface, it will diverge from a point after reflexion.*

Let Q be the origin of a pencil whose axis QA is incident directly on a plane reflecting surface AR'. Then after reflexion the pencil will diverge from a point q in QA

produced, at a distance $Aq = AQ$ (Art. 16). If now we suppose the whole pencil to be removed with the exception of the oblique pencil QRR', the course of this portion of the pencil will remain unaltered, or the oblique pencil will after reflexion diverge from q. Similarly if an oblique pencil *converging* to q be reflected at the plane surface, it will after reflexion converge to Q.

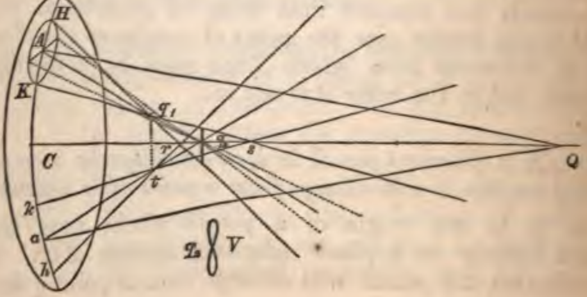


COR. Since $\angle RQR' = \angle RqR'$, the degree of divergence of the reflected pencil is the same as that of the incident pencil.

61. Other cases of oblique reflexion or refraction are not so simple as that of the preceding Article; we shall now shew that if the pencil be *small*, it will after reflexion or refraction *converge to* or *diverge from* two very small straight lines, perpendicular to one another and in different planes, so that the direction of every ray passes through each of these straight lines.

62. *To explain the formation of focal lines, when a small oblique pencil is reflected at a spherical surface, or refracted at a plane or spherical surface.*

Let QC be the axis of a pencil incident directly at C on a spherical reflecting surface, *or on a plane or spherical re-*



which meet [illegible] rays in the same [illegible] Suppose *kr* [illegible] perpendicular [illegible]

Now [illegible] AH, when K [illegible] fore for some [illegible]

(ii) If *K* [illegible] circle with [illegible] *n* [illegible] *AF*, is the [illegible] passes. For a [illegible] larger in consequence [illegible] cular section [illegible] diverging cone [illegible] *Aberration* of the [illegible]

Obs. In the preceding figure the aberration is supposed to be *towards* the surface; the student will have little difficulty in drawing a corresponding figure for the case when the aberration is *from* the surface.

In the following calculation of the position and dimensions of this circle, the cube and higher powers of the latitude of the pencil will be neglected, as has been done in former cases.

56. *To calculate the position and dimensions of the Least Circle of Aberration after direct reflexion or refraction at a plane or spherical surface.*

Let $AK=y'$, $AH=y$, $r=$ rad. of spherical surface AK (fig. Art. 55). Draw KM perpendicular to AF.

$$\therefore\ KM = r\sin\frac{y'}{r} = y',$$

$$AM = r\text{ vers.}\frac{y'}{r} = \frac{y'^2}{2r},\ \text{nearly}.$$

$$\therefore\ mt = \frac{1}{4} \cdot \frac{Fr}{Ar}\, y \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

$$rm = \frac{1}{4} \cdot Fr \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)};$$

two equations define the position and magnitude of the circle of aberration.

other words, the distance of the circle of least aberration the geometrical focus is three-fourths of the longitudinal ation of the pencil, and its radius is one-fourth of the teral aberration of the extreme ray.

pencil diverges not from *a point*, but from two focal lines at q_1, q_2, these lines being at right angles to each other and not in the same plane. Hence the image of Q seen by E is more or less indistinct, and is seen in the direction of a tangent drawn from the eye to the caustic of Q in the plane EQC. Since the *circle of least confusion* is the nearest approach to a *point* which such a small oblique pencil admits of, we may perhaps regard the circle of least confusion as the image of Q, —and when an object of *finite* size is viewed by reflexion *or refraction*, we may consider the visible image to be the locus of the circles of least confusion of small oblique pencils emanating from consecutive points of the object; the position of any one of these being determined as above. These circles of confusion will overlap each other, and the image consequently will be more or less confused and indistinct.

We may regard the comparative size of the circles of least confusion in different cases as a measure of the comparative indistinctness of the visible image.

It will sometimes happen that from an eye in a given position more than one tangent can be drawn to the caustic curve in the plane QCE; in such cases there will be more than one visible image.

In the valuable optical diagrams published by *Engel and Schellbach*—(Halle: *first series*, 1850, *second series*, 1856)—the locus of the *primary focus* is taken as the visible image; this can only be received as an *approximate* position of the image; but nevertheless the student will derive great advantage from consulting them. A simplified and less expensive edition of the first series has been published by the *Rev. W. B. Hopkins, late Tutor of St Catharine's College.*

It will be a matter of convenience frequently for us to treat the primary focus as the point from which the visual pencil diverges, and regard the visual image as the locus of the primary foci of the small oblique pencils received by the eye from consecutive points of the object.

[illegible]

[illegible]

e figure [illegible]

stead of the conical [illegible] us, let us now consider [illegible] it *HK*, by which we [illegible] *l* whose axis after reflexion [illegible]

ion of every reflected *or refracted* ray now [illegible] ass through some point of a small circular arc [illegible] ay approximately be regarded as a straight line [illegible] to the plane *QCA*. This line is called the [illegible] *Line*, and the point q_1 *the Primary Focus*.

ection of the pencil by a plane through [illegible] tangent plane of the [illegible]

Now $\delta\phi = HQA = \dfrac{An}{AQ}$ approximately $= \dfrac{AH\cos\phi}{u}$,

$$\delta\phi' = Hq_1A = \frac{At}{Aq_1} = \frac{AH\cos\phi'}{v_1}.$$

If μ be the index of refraction from the first medium into the second,

$$\sin\phi = \mu\sin\phi' \text{ (Art. 9)},$$

and by differentiating this we get the relation between the small corresponding variations $\delta\phi$, $\delta\phi'$, viz.

$$\cos\phi\,.\,\delta\phi = \mu\cos\phi'.\,\delta\phi';$$

therefore substituting the above values of $\delta\phi$, $\delta\phi'$,

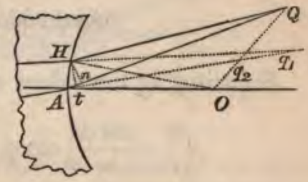
$$\frac{\mu\cos^2\phi'}{v_1} - \frac{\cos^2\phi}{u} = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

Again, $$\frac{u}{v_2} = \frac{AQ}{Aq_2} = \frac{\sin\phi'}{\sin\phi} = \frac{1}{\mu};$$

$$\therefore \frac{\mu}{v_2} - \frac{1}{u} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

The results (i), (ii) give respectively the distances of the *primary* and *secondary* foci from A.

68. *A small pencil is obliquely refracted at a spherical surface; to find the distances of the foci from the point of incidence of the axis.*



Let Q be the origin of a small pencil whose axis QA is incident obliquely at A on a spherical refracting surface whose centre is O. Let QO cut the direction of the axis QA after refraction in q_2, *the secondary focus.* Let QH be any ray in the primary plane whose direction after refraction cuts Aq_2 in q_1, *the primary focus.* Join AO, HO, and draw

Hn perpendicular to AQ and Ht perpendicular to Aq_1. L
OH, AQ intersect in K, and OH, Aq_1 in T.

Let $AQ=u$, $Aq_1=v_1$, $Aq_2=v_2$, $AO=r$, $AOq_2=\theta$,

$$\left.\begin{matrix}\phi\\ \phi+\delta\phi\end{matrix}\right\}\text{ angles of incidence of }\left\{\begin{matrix}QA\\ QH\end{matrix}\right.,$$

$$\left.\begin{matrix}\phi'\\ \phi'+\delta\phi'\end{matrix}\right\}\text{ .... refraction ....}$$

Then $$\angle HQA=\frac{Hn}{HQ}\text{ approximately,}$$

$$=\frac{AH\cos\phi}{u}\text{ ultimately;}$$

$$Hq_1A=\frac{AH\cos\phi'}{v_1}.$$

But $$\delta\phi=QHO-QAO=\angle K-HQA-(\angle K-AOH)$$

$$=AOH-HQA=\frac{AH}{r}-\frac{AH\cos\phi}{u},$$

and $$\delta\phi'=q_1HO-q_1AO=\angle T-Hq_1A-(\angle T-AOH)$$

$$=AOH-Hq_1A=\frac{AH}{r}-\frac{AH\cos\phi'}{v_1}.$$

And from the relation $\sin\phi=\mu\sin\phi'$, we get by differentiation

$$\cos\phi\,.\,\delta\phi=\mu\cos\phi'.\,\delta\phi'.$$

Substituting the values of $\delta\phi$, $\delta\phi'$, found above, we have

$$\left(\frac{1}{r}-\frac{\cos\phi}{u}\right)\cos\phi=\left(\frac{1}{r}-\frac{\cos\phi'}{v_1}\right)\mu\cos\phi',$$

or $$\frac{\mu\cos^2\phi'}{v_1}-\frac{\cos^2\phi}{u}=\frac{\mu\cos\phi'-\cos\phi}{r}\text{............(i).}$$

Ha perpendicular to AQ and Hb perpendicular to A
OH, AQ intersect in K, and OH, Aq_1 in T.

Let $AQ = u$, $Aq_1 = v_1$, $Aq_2 = v_2$, $AO = r$, AOq

$$\left.\begin{matrix}\phi\\ \phi+\delta\phi\end{matrix}\right\}\text{ angles of incidence of }\left\{\begin{matrix}QA\\ QH\end{matrix}\right.,$$

$$\left.\begin{matrix}\phi'\\ \phi'+\delta\phi'\end{matrix}\right\}\text{ .... refraction ....}$$

Then
$$\angle HQA = \frac{Ha}{HQ}\text{ approximately,}$$

$$= \frac{AH\cos\phi}{u}\text{ ultimately}$$

$$Hq_1A = \frac{AH\cos\phi'}{v_1}.$$

But $\delta\phi = QHO - QAO = \angle K - HQA - (\angle K$

$$= \angle OH - HQA = \frac{AH}{r} - \frac{AH\cos\phi}{u}$$

and $\delta\phi' = q_1HO - q_1AO = \angle T - Hq_1A - (\angle$

$$= \angle OH - Hq_1A = \frac{AH}{r} - \frac{AH\cos\phi}{v_1}$$

And from the relation $\sin\phi = \mu\sin\phi'$,
tiation

$$\cos\phi\,.\,\delta\phi = \mu\cos\phi'\,.\,\delta\phi$$

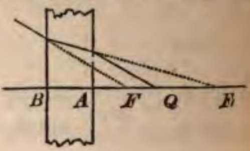
Substituting the values of $\delta\phi$, $\delta\phi'$, fo

$$\left(\frac{1}{r} - \frac{\cos\phi}{u}\right)\cos\phi = \left(\frac{1}{r} - \frac{\cos}{v_1}\right.$$

or
$$\frac{\mu\cos^2\phi'}{v_1} - \frac{\cos^2\phi}{u} = \frac{\mu\cos\phi' -}{r}$$

85. *To determine the geometrical focus of a pencil after direct refraction through a plate.*

Let Q be the origin of a pencil whose axis QAB passes directly through a refracting plate. Let F_1, F be the geometrical foci of the pencil after refraction at the first and second surfaces respectively.



Let $AQ=u$, $BF=v$, $AB=t$.

From the first refraction (Art. 19)

$$AF_1=\mu u.$$

Now F_1 being regarded as an origin, a pencil diverging from F_1 after refraction at the second surface, has F for its geometrical focus. Hence if the course of the pencil be supposed reversed, a pencil converging to F would after refraction into the plate have F_1 for its geometrical focus; (Art. 79)

$$\therefore\ BF_1=\mu\,.\,BF,\ \text{ or }\ AF_1+t=\mu v;$$

$$\therefore\ t=\mu v-\mu u,\quad \text{ or }\ v=u+\frac{t}{\mu},$$

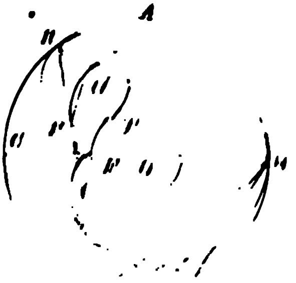
which determines the position of F.

Note. Since $BF-BQ=-\dfrac{\mu-1}{\mu}\,.\,t$, it follows that an object appears nearer when viewed through a plate denser than the surrounding medium.

86. When a small pencil is refracted obliquely through a portion of medium bounded by two given surfaces, there is an apparent difficulty at the second refraction in consequence of neither the incident nor the refracted pencil having then a *point* of divergence or convergence.

In the primary plane, however, the pencil on account of its smallness may after each refraction be regarded as converging to or diverging from a point, and we may thus apply—with erence to this plane—propositions which suppose the whole

RP the direction of *QR* after
exion. join *O'P*, *PT*.



Then

$$O'T = 2PRT = 2QRO = TOF,$$

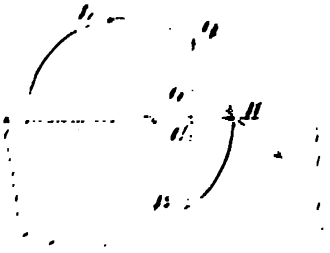
$$O'T = OT;$$

$$\therefore \text{arc } TP = \text{arc } TF.$$

Hence it will readily appear
t the reflected rays all touch
e epicycloid traced out by the
nt *P* of the circle *RPT*, as this circle rolls upon *BO*, and
is epicycloid is the caustic curve required; it has a cusp
F, the geometrical focus of rays reflected at *O*, and touches
e mirror at *Q*.

73. *Rays diverging from a point are refracted at a plane
rface, the caustic is the evolute of a conic section.*

(i) Suppose the refraction to take place from a denser into a rarer medium. Let *QR* be any ray diverging from *Q* and incident at *R*, on the plane surface. Draw *QCB* perpendicular to the surface, and make $CB = QC$. Describe a circle about *QRB*, and let *LR* be the direction of the refracted ray.



Join *QL*, *LB*; *O* the point of intersection of *QC*, *LR*.

Since *LR* bisects ∠ *QLB*, we have $\dfrac{OB}{OQ} = \dfrac{LB}{LQ}$;

$$\therefore \frac{QB}{OQ} = \frac{BL + QL}{QL},$$

$$\therefore \frac{QB}{BL + QL} = \frac{OQ}{QL} = \frac{\sin QLO}{\sin QOL}.$$

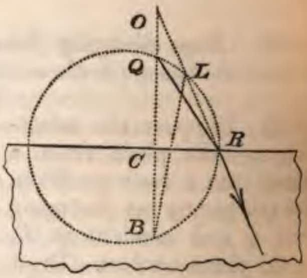
And $QLR = QBR = RQB = \phi$, the angle of incidence of QR,

$$QOL = \phi' = \angle \text{ of refraction of } QR;$$

$$\therefore \frac{QB}{BL + QL} = \frac{\sin\phi}{\sin\phi'} = \mu, \text{ (where } \mu < 1).$$

Hence $BL + QL = \dfrac{BQ}{\mu}$ = a constant quantity, and the locus of L is an ellipse of which Q, B are the foci,—and LR is a normal at L to the locus of L, since it bisects the angle between the focal distances; that is, the caustic curve is the evolute of the ellipse whose foci are Q, B and axis major

$$= \frac{BQ}{\mu}.$$



(ii) If the refraction takes place from a rarer into a denser medium, let LR be the direction of the refracted ray, produce LR to meet CQ produced in O,—then it may be shewn that LO bisects the exterior angle QLB, and

$$\frac{BL}{QL} = \frac{BO}{QO};$$

$$\therefore \frac{BL - QL}{QL} = \frac{BQ}{QO};$$

$$\therefore \frac{QB}{BL - QL} = \frac{QO}{QL} = \frac{\sin OLQ}{\sin LOQ} = \frac{\sin\phi}{\sin\phi'} = \mu, \ (\mu > 1).$$

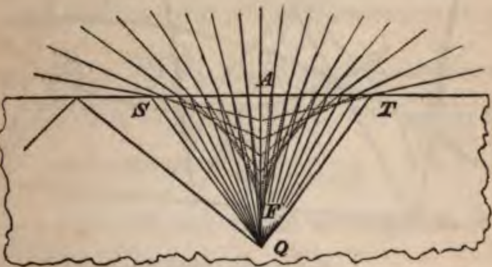
Hence the locus of L is a hyperbola, and OL is the direction of the normal at L; that is, the caustic curve is the evolute of a hyperbola whose foci are Q, B, and axis major

$$= BL - QL$$

$$= \frac{QB}{\mu}.$$

. *Illustration.*

ıus, if Q be a radiant point, the rays from which are ;ed at a plane surface ST into a rarer medium, the ent rays will be tangents to a *virtual* caustic SF, FT, is a portion of the evolute of an ellipse, one cusp of

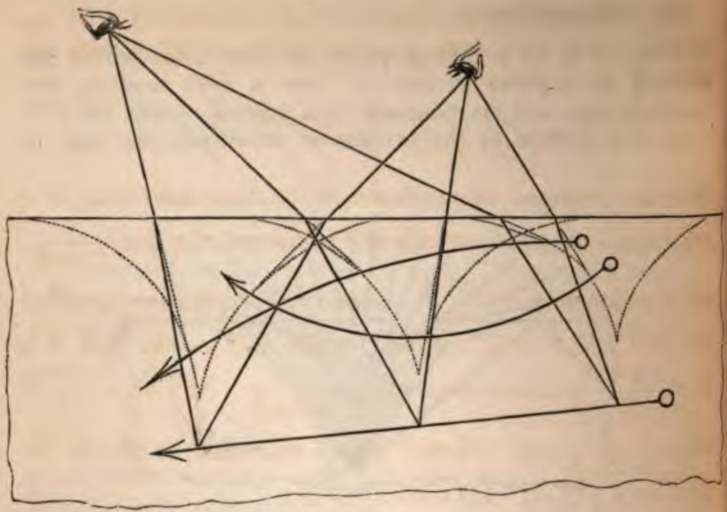


ı is at F the geometrical focus of Q. The branches ɛ caustic touch the plane surface of the medium at S, T, s determined by the condition that the rays QS, QT are ɛnt at the critical angle. (Art. 80.)

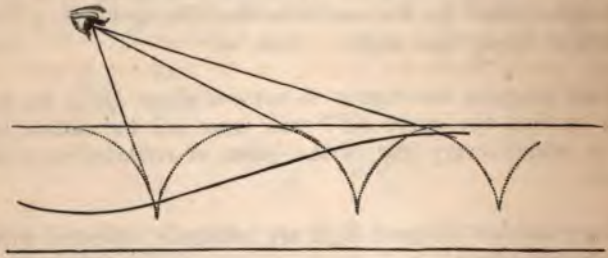
' we suppose the figure to revolve about QFA the axis e surface, the curve SFT will trace out the caustic surto which every ray of the beam of refracted rays is a ınt.

'ays incident beyond S, T are internally reflected within enser medium.

'he caustic just obtained enables us to illustrate the mation of the visible image of an object situated in a ɜr medium than that in which the eye is situated, the a being bounded by a plane. The dotted curves in the e (—see the next page—) are intended to represent the ics of three different points of the object, and to avoid ısion, only the *axes* of the pencils which the eye receives rawn.



Again, suppose a surface of still water with a level horizontal bottom not very deep; the bottom will not appear a



plane, but will seem to rise on all sides, being shaped something like a basin below the eye, and tending at distant points towards the surface of the water as an asymptotic plane.

75. *If a small pencil of diverging rays be reflected at a concave spherical mirror, to find the limits of the distance of the origin from the point of incidence in order that the reflected pencil may converge to, or diverge from, both the focal lines.*

The distances of the primary and secondary foci of the eflected pencil from the point of incidence of its axis are given by the equations

$$\frac{1}{v_1} = \frac{2}{r\cos\phi} - \frac{1}{u}, \quad \frac{1}{v_2} = \frac{2\cos\phi}{r} - \frac{1}{u}.$$

Now the reflected pencil *converges* to each focal line, if v_1 and v_2 are both positive; this will be the case if u be $> \frac{r}{2\cos\phi}$, and consequently $> \frac{r\cos\phi}{2}$, ... and it *diverges* from each line f v_1 and v_2 are both negative, which will be the case if u be $< \frac{r\cos\phi}{2}$, and consequently $< \frac{r}{2\cos\phi}$. If u be $> \frac{r\cos\phi}{2}$ and $< \frac{r}{2\cos\phi}$, the reflected pencil converges to one of the focal lines and diverges from the other.

If the rays reflected in the primary plane are parallel or v_1 be infinite, then $u = \frac{r\cos\phi}{2}$ and $v_2 = -\frac{r\cos\phi}{2\sin^2\phi}$; the atter is the polar equation to the locus of the secondary focus, when the origin assumes different positions in the same primary plane, so that the rays in the primary plane may be parallel.

Now

and $OT = 2$

We may
circle TF, t
metrical foc
epicycloid F
RPT is a rig
the path of
i. e. each ref
therefore the

This cau
mirror at A
we should g
hemispheric

Obs. T
shewn exper
—(a piece
cave form.
exposed to t
passes nearl
be seen trac
defined line ;
and the light
inwards. If
catacaustics w
&c. will be see
the surface of
in the sunshin

72. *To fin*
curve is a circ
in its circumfer

Let QR be
circumference
Q. Join OR,
describe a circle
so that its radiu

Since $i=\phi'+\psi'$ the lower sign is inadmissible.

Hence $\phi'=\psi'$,

and therefore $\phi=\psi$ is the only result.

Further, differentiating (i), (ii), (iii), (iv),

$$\frac{d^2D}{d\phi^2}=\frac{d^2\psi}{d\phi^2}, \quad \frac{d^2\phi'}{d\phi^2}+\frac{d^2\psi'}{d\phi^2}=0;$$

$$\cos\psi\,.\,\frac{d^2\psi}{d\phi^2}-\sin\psi\,.\left(\frac{d\psi}{d\phi}\right)^2=\mu\cos\psi'\,\frac{d^2\psi'}{d\phi^2}-\mu\sin\psi'\left(\frac{d\psi'}{d\phi}\right)^2$$

$$-\sin\phi=\mu\cos\phi'\,.\,\frac{d^2\phi'}{d\phi^2}-\mu\sin\phi'\,.\left(\frac{d\phi'}{d\phi}\right)^2.$$

Putting $\phi=\psi$, $\phi'=\psi'$, we get for the value of $\frac{d^2D}{d\phi^2}$,

$$\frac{d^2D}{d\phi^2}=\frac{2(\mu^2-1)\sin\phi}{\mu^2\cos^2\phi'\cos\phi}=\text{a positive quantity.}$$

Hence $\phi=\psi$ makes D a *minimum*.

That is, when the angle of incidence is equal to the angle of emergence, the deviation is less than in any other case.

94. We give another method of obtaining the condition of minimum deviation, which does not require the use of the *Differential Calculus.*

With our usual notation

$$D=\psi+\phi-i, \quad i=\phi'+\psi',$$
$$\sin\phi=\mu\sin\phi', \quad \sin\psi=\mu\sin\psi'.$$

Eliminating ϕ', ψ' we get

$$\sin\phi=\mu\sin\phi'=\mu\sin(i-\psi')=\mu\sin i\cos\psi'-\cos i\sin\psi;$$

$$\therefore\ (\sin\phi+\cos i\sin\psi)^2+(\sin\psi\sin i)^2=\mu^2\sin^2 i,$$

r $$\sin^2\phi+2\cos i\sin\phi\sin\psi+\sin^2\psi=\mu^2\sin^2 i.$$

reflexion at any number of plane surfaces in succession, if the lines of intersection of the surfaces be all parallel.

A familiar instance of rays so reflected is afforded by the *Kaleidoscope.*

Successive Refractions at Plane Surfaces.

79. In examining the effect of successive refractions on a pencil, the following two considerations are employed:—

(i) The geometrical focus of a direct pencil being the point of ultimate intersection of any refracted ray with the axis, the pencil after one refraction may ultimately be considered to diverge from or to converge to this point as an origin, the pencil being supposed a very small one.

(ii) If a ray be reflected or refracted in any manner in passing from one point to another, it is assumed that it might pass by the same course reversed from the latter point to the former. This is in accordance with a general law in Optics, that the visibility of two points from one another is mutual; in other words, if a ray of light proceeding from A arrives by any course at B however often reflected or refracted, a ray can also arrive at A from B by retracing precisely the same course in the opposite direction.

Hence when a pencil is emerging from a refracting medium, we may when convenient reduce this case to the more familiar one of a pencil entering a refracting medium, by supposing the course of each ray reversed.

Critical Angle.

80. If ϕ be the angle of incidence of a ray of light, and ϕ' its angle of refraction into a denser medium, μ the refractive index between the media, then

$$\sin\phi = \mu\sin\phi', \text{ or, } \sin\phi' = \frac{1}{\mu}\cdot\sin\phi.$$

Now μ being >1, (Art. 9), $\sin\phi'$ is <1, and this equation ves a real angle of refraction for any given angle of incihe equivalent form in $r\ \theta$ is

$$\frac{d\theta}{dr}=\frac{C}{r\sqrt{(\mu^2r^2-C^2)}}.$$

Note. The form (iii) might have been obtained more readily,—for if ϕ' be the angle of refraction at P, $\frac{\mu'}{\mu}$ the refractive index,

$$\sin\phi=\frac{\mu'}{\mu}\sin\phi'\,;\quad\therefore\ \mu'r\sin\phi'=\mu r\sin\phi\,;$$

$$\therefore\ \mu'p'=\mu p\,;\quad\therefore\ d(\mu p)=0,\quad\text{or}\quad \mu p=C\ \text{a constant.}$$

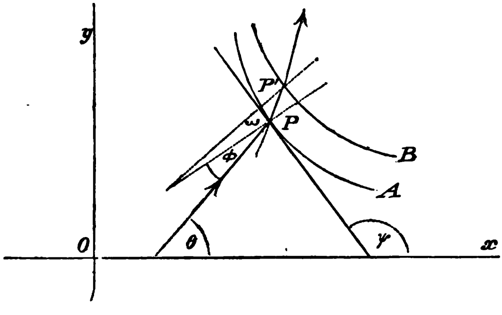
Ex. 1. Suppose $\mu\propto\frac{1}{r}$, the path is an equiangular spiral.

Ex. 2. Suppose $\mu\propto\frac{1}{\sqrt{r}}$, the path is a parabola, focus C.

122. *A ray is propagated through a medium of variable density which is symmetrical with respect to the plane of xy —(which is also the plane of the ray),—the differential equation to the path is*

$$\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}=\frac{1}{\mu}\left(\frac{d\mu}{dy}-\frac{d\mu}{dx}\frac{dy}{dx}\right),$$

where $\mu=f(xy)$ *is the refractive index at any point* xy, *and* $\frac{d\mu}{dx}$, $\frac{d\mu}{dy}$ *are the partial differential coefficients of* μ.



a result which shews that (μ being > 1) the primary or secondary focus is the nearer of the two to the prism, according as ϕ is $>$ or $< \psi$.

Cor. 2. If the pencil at emergence diverge from a *point*, i.e. if $v_1 = v_2$—we must have

$$\frac{\cos^2\phi'}{\cos^2\phi}\cdot\frac{\cos^2\psi}{\cos^2\psi'} = 1, \text{ and } \therefore\ \phi = \psi,$$

which result is also the condition of minimum deviation. In this case the emergent pencil diverges from a point at the same distance as its origin from the edge of the prism, and the degree of divergence remains unaltered.

This result is of importance in Newton's experiment hereafter described.

Successive Refractions at Spherical Surfaces.

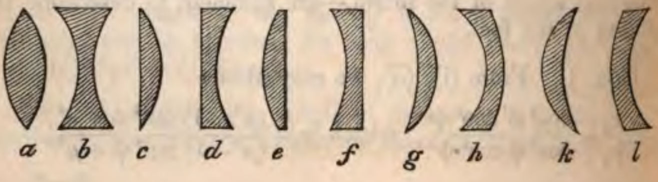
97. *Def.* A *lens* is a portion of a refracting medium bounded by two surfaces of revolution which have a common axis, called *the axis of the lens.*

Obs. The bounding surfaces of a lens will, unless the contrary be expressed, be considered spherical—under which denomination plane surfaces are included as a particular case when the radius of the sphere is infinite.

If the surfaces of revolution do not meet, the additional surface which is required as a boundary to the lens round the edge, will be a cylindrical one, having its axis coincident with that of the lens.

98. Lenses are distinguished by different names, according to the nature of their surfaces.

Thus in the figure:—



whence
$$\frac{\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2}=\frac{1}{\mu}\left(\frac{d\mu}{dy}-\frac{d\mu}{dx}\cdot\frac{dy}{dx}\right),$$

which is the differential equation to the path of the ray.

123. We may easily obtain the same result from the principle that when a ray passes through any medium then $\Sigma(\mu\,.\,ds)$ = a minimum, or

$$\int\mu\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,.\,dx = \text{a minimum.}$$

With the usual notation of the *Calculus of Variations* since μ is a function of x and y, $=\phi(xy)$ suppose,

$$V=\phi(xy)\sqrt{(1+p^2)} \text{ and } N-\frac{dP}{dx}=0\text{.........}(\text{i}).$$

Now $\quad N=\dfrac{d\mu}{dy}\sqrt{(1+p^2)},\;\; P=\dfrac{\mu p}{\sqrt{(1+p^2)}}$;

therefore (i) gives

$$\sqrt{(1+p^2)}\,.\,\frac{d\mu}{dy}=\frac{d}{dx}\left\{\mu\frac{p}{\sqrt{(1+p^2)}}\right\}$$

$$=\left(\frac{d\mu}{dx}+\frac{d\mu}{dy}p\right)\frac{p}{\sqrt{(1+p^2)}}+\mu\frac{\dfrac{dp}{dx}}{(1+p^2)^{\frac{3}{2}}};$$

$$\therefore\; \frac{\dfrac{dp}{dx}}{1+p^2}=\frac{1}{\mu}\left(\frac{d\mu}{dy}-\frac{d\mu}{dx}\cdot\frac{dy}{dx}\right),$$

the same as before.

If ρ be the radius of curvature at the point xy of the path the preceding result can be put in the form

$$\frac{\mu}{\rho}=\frac{d\mu}{dx}\cdot\frac{dy}{ds}-\frac{d\mu}{dy}\cdot\frac{dx}{ds}.$$

124. If $\mu=\phi(xyz)$ be the absolute refractive index in the most general case of a medium of varying density, from

of the lens respectively, lines being regarded as positive when measured in a direction opposite to that of the incident light.

From the refraction at the first surface

$$\frac{\mu}{AF_1} - \frac{1}{u} = \frac{\mu - 1}{r}. \quad \text{(Art. 27)} \ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

Now a pencil converging to F would have F_1 for its geometrical focus after refraction at B, (Art. 79);

$$\therefore \frac{\mu}{BF_1} - \frac{1}{v} = \frac{\mu - 1}{s} \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)};$$

therefore, if the thickness be neglected, or AF_1 regarded as $= BF_1$, we get

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

which result determines the position of F.

Note. The points Q and F are called conjugate foci, with respect to the lens.

100. Cor. 1. If t be the thickness of the lens at the axis, since

$$t = BF_1 - AF_1,$$

we might from (i) and (ii) obtain

$$\frac{1}{\dfrac{1}{v} + \dfrac{\mu - 1}{s}} - \frac{1}{\dfrac{1}{u} + \dfrac{\mu - 1}{r}} = \frac{t}{\mu},$$

a relation which is strictly accurate, and which may be used when the thickness is too considerable to allow of any of its powers being neglected.

Cor. 2. If the thickness be sensible, but small, and $= t$, since

$$\frac{\mu}{BF_1} = \frac{\mu}{AF_1 + t} = \frac{\mu}{AF_1\left(1 + \dfrac{t}{AF_1}\right)} = \frac{\mu}{AF_1}\left(1 - \frac{t}{AF_1}\right) \text{ nearly},$$

[illegible]

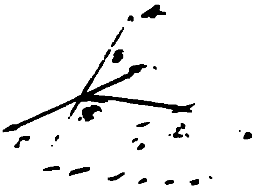
[illegible]

[illegible] $< \frac{\pi}{2}$.

[illegible] the angle of incidence [illegible] $\mu < 1$.

[illegible] through the prism [illegible] of the prism [illegible] medium.

Fig. 6.



[illegible] deviation [illegible] tangent [illegible] therefore the [illegible]

[illegible]

[illegible] the angle [illegible]

[illegible] condition [illegible]

$$\text{whence } \frac{1}{f'}=\frac{1}{f}-\frac{\dfrac{\mu-1}{\mu}\cdot\dfrac{t}{r}}{\dfrac{r}{\mu-1}+\dfrac{t}{\mu}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)},$$

a relation between f and f'.

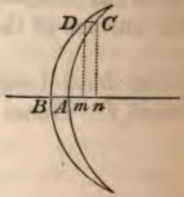
If we neglect the square of $\frac{t}{r}$, then the approximation deduced from (iii), gives

$$f'=f+\left(\frac{\mu-1}{r}\right)^2\cdot\frac{tf^2}{\mu};$$

which accords with the result of Art. 100, Cor. 2.

Note. It will appear from the preceding, that when the thickness of a lens is not neglected the focal length does not remain the same when the order of the surfaces at which the refractions take place, is reversed.

102. We will next examine in what cases the focal length of a lens is positive or negative.



Let $\tau = CD$ the thickness of a lens measured parallel to the axis at a distance $Cn=y$ from the axis, $t=AB$ the value of τ when $y=0$, i.e. at the axis; r, s the radii of the first and second surfaces.

Then $\quad An=\dfrac{y^2}{2r}$, very nearly,

$$Bm=\frac{y^2}{2s},$$

and $\quad Bm+\tau=t+An;$

$$\text{i.e. } \tau-t=An-Bm=\frac{y^2}{2}\cdot\left(\frac{1}{r}-\frac{1}{s}\right)=\frac{y^2}{2(\mu-1).f}.$$

Hence f is positive or negative according as τ is $>$ or $< t$, i.e. according as the lens is thinnest or thickest at the axis.

of each surface on which the ray is incident—meet the sphere in A, B: and a radius parallel to the intersection of the two surfaces, in I.

Let

$$\left.\begin{array}{l} PA=\phi \\ QA=\phi' \end{array}\right\} \text{the angles of incidence and refraction at 1st surface,}$$

$$\left.\begin{array}{l} QB=\psi' \\ RB=\psi \end{array}\right\} \text{.. 2nd surface,}$$

$AB=i$ the angle between the two surfaces,

$PAB=\theta_1$, $QBA=\theta_2$ the angles which the planes of *first* and *second* refraction make with the principal plane, *i.e.* the plane perpendicular to each surface,

$\omega=AQB=$ the angle between the planes of *first* and *second* refraction,

$\mu\,\mu'$ the indices of refraction from the *first* to the *second*, and from the *second* to the *third* medium,

$D=PR$ the deviation of the ray;—then we have the relations

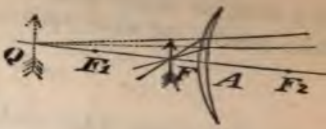
$$\frac{\sin\omega}{\sin i}=\frac{\sin\theta_1}{\sin\psi'}=\frac{\sin\theta_2}{\sin\phi'},$$

$$\left.\begin{array}{l} \cos i=\cos\phi'\cos\psi'+\sin\phi'\sin\psi'\cos\omega, \\ \cos D=\cos(\phi-\phi')\cos(\psi-\psi')+\sin(\phi-\phi')\sin(\psi-\psi')\cos\omega \\ \text{and} \\ \sin\phi=\mu\sin\phi', \quad \sin\psi'=\mu'\sin\psi. \end{array}\right\}.$$

These equations are sufficient to determine the circumstances of the transmission of the ray, when the quantities given render the problem determinate.

Cor. If the first and third medium are the same, the case becomes that of refraction through a prism, and $\mu\,.\,\mu'=1$. In this case we can shew that $IP=IR$.

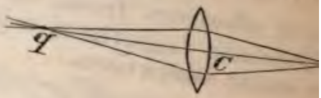
(iv) When u becomes negative or the incident pencil is convergent, the emergent pencil is convergent,—and as Q moves off from A to an infinite distance, F moves from A up to F_1.



Similarly the change of relative position may be tr when the lens is *concave*.

108. *Method of determining the focal length of a co lens practically.*

Let a small bright object Q be placed further from lens than its principal focus, q the real image of Q on the opposite side of the lens,



$$CQ = u,$$

f = numerical focal length,

$$Qq = x, \text{ then } \frac{1}{Cq} + \frac{1}{CQ} = \frac{1}{f},$$

$$\text{or } \frac{1}{x-u} + \frac{1}{u} = \frac{1}{f};$$

$$\therefore fx = ux - u^2;$$

$$\therefore u = \frac{x \pm \sqrt{\{x(x-4f)\}}}{2}.$$

It appears from this result that the least positive value x is $4f$: hence if the image of Q formed at q be received o screen, and the lens and screen be moved backward and f ward till the distance of the image from Q is the least possib the focal length of the lens—without reference to algebraic si —is one-fourth of this distance.

If the lens be concave, let it be placed in contact with convex lens whose focal length f' is such that the focal leng F of the combination may be negative, the axes of the tw lenses being coincident.

Then if f' and F be each determined in the precedin f is to be formed from the formula (*algebraic*)

[illegible]ons on the pencil independent[illegible] s length [illegible]
[illegible]he prism.

Let Q be the [illegible]
[illegible] whose axis is [illegible]
[illegible] in a principal [illegible]
[illegible] in direction QA [illegible]
the primary and secondary [illegible]
[illegible] the first refraction, [illegible]
emergence.

$AQ =$ [illegible] $=$ [illegible] $Aq_1 =$ [illegible]

ϕ the angles of incidence and refraction [illegible]
ϕ' .... emergence and incidence [illegible]

From the first refraction,

$$\frac{\mu \cos^2 \phi}{\cos^2 \phi}$$ [illegible]

Now the pencil emanating [illegible]
[illegible]verge [illegible] that plane after [illegible]
[illegible] suppose [illegible] reversed [illegible]
[illegible]verge [illegible] the primary plane [illegible]
[illegible]

$\therefore AQ$ [illegible]

Similarly [illegible]

[illegible]
[illegible]fracting [illegible]
[illegible]osition [illegible]

Cor. [illegible]

of the lens respectively, lines being regarded as positive measured in a direction opposite to that of the incident l

From the refraction at the first surface

$$\frac{\mu}{AF_1} - \frac{1}{u} = \frac{\mu - 1}{r}. \quad \text{(Art. 27)} \ldots\ldots\ldots\ldots\ldots$$

Now a pencil converging to F would have F_1 for its metrical focus after refraction at B, (Art. 79);

$$\therefore \frac{\mu}{BF_1} - \frac{1}{v} = \frac{\mu - 1}{s} \ldots\ldots\ldots\ldots\ldots\ldots(\text{i}$$

therefore, if the thickness be neglected, or AF_1 regar $= BF_1$, we get

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

which result determines the position of F.

Note. The points Q and F are called conjugate foci, respect to the lens.

100. Cor. 1. If t be the thickness of the lens at the a since

$$t = BF_1 - AF_1,$$

we might from (i) and (ii) obtain

$$\frac{1}{\dfrac{1}{v} + \dfrac{\mu - 1}{s}} - \frac{1}{\dfrac{1}{u} + \dfrac{\mu - 1}{r}} = \frac{t}{\mu},$$

a relation which is strictly accurate, and which may be u when the thickness is too considerable to allow of any of powers being neglected.

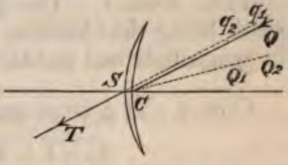
Cor. 2. If the thickness be sensible, but small, and since

$$\frac{\mu}{BF_1} = \frac{\mu}{AF_1 + t} = \frac{\mu}{AF_1\left(1 + \dfrac{t}{AF_1}\right)} = \frac{\mu}{AF_1}\left(1 - \frac{t}{AF_1}\right) \text{ nearl}$$

For if R and S be the points of incidence and emergence the surfaces at these points being parallel, the case is the same as that of a ray refracted through a plate whose surfaces touch the lens at R and S.

When a pencil is refracted obliquely through a lens there will be an important difference produced, according as the pencil is refracted *centrically* or *excentrically*, i. e. according as the direction of its axis between the two refractions does or does not pass through the centre of the lens.

112. *When a small pencil is obliquely and centrically refracted through a thin lens, to find the distances of the foci of the emergent pencil from the centre of the lens.*



Let Q be an origin of a small pencil whose axis $QCST$ is refracted obliquely and centrically through a lens,—C being the point where the axis of the lens meets its first surface, which point is the centre of the lens if the thickness of the lens be neglected (Art. 109). Let Q_1, Q_2 be the primary and secondary foci of the pencil after one refraction, q_1, q_2 the primary and secondary foci of the emergent pencil.

Let $CQ = u$, $Sq_1 = v_1$, $Sq_2 = v_2$, r, s the radii of the first and second surfaces of the lens, ϕ, ϕ' the angles of incidence and refraction of the axis QC at C, and consequently the angles of emergence and incidence at S.

From refraction at the first surface

$$\frac{\mu \cos^2 \phi'}{CQ_1} - \frac{\cos^2 \phi}{u} = \frac{\mu \cos \phi' - \cos \phi}{r}, \text{ (Art. 68).}$$

Now the pencil emanating in the primary plane from Q_1 diverges in that plane after the second refraction from q_1; hence if we suppose its course reversed, a pencil converging to q_1 will in the primary plane converge to Q_1 after refraction into the lens;

$$\therefore \frac{\mu \cos^2 \phi'}{CQ_1} - \frac{\cos^2 \phi}{v_1} = \frac{\mu \cos \phi' - \cos \phi}{s},$$

the points C, S being regarded as coincident;

The focal length of the lens being given, the value of v for a proposed value of u is known. Hence we must have

$$\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)-\left(\frac{1}{s}-\frac{1}{v}\right)^2\left(\frac{1}{s}-\frac{\mu+1}{v}\right)=\text{a minimum},$$

whilst
$$(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right)=\frac{1}{f}=\frac{1}{v}-\frac{1}{u}.$$

In consistence with these two latter conditions, let us assume

$$\frac{1}{v}=\frac{\alpha+1}{2f},\quad \frac{1}{u}=\frac{\alpha-1}{2f},$$

also
$$\frac{1}{r}=\frac{x+1}{2(\mu-1)f},\quad \frac{1}{s}=\frac{x-1}{2(\mu-1)f},$$

so that x is the only variable quantity,—we get by substitution,

$$\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\frac{1}{r}-\frac{\mu+1}{u}\right)$$

$$=\frac{1}{8f^3(\mu-1)^3}\{x-(\mu-1)\,\alpha+\mu\}^2\{x-(\mu^2-1)\,\alpha+\mu^2\}\;\ldots\text{(i)}.$$

Now we pass from $\left.\begin{array}{l}\frac{1}{r}\text{ to }-\frac{1}{s}\\[4pt]\frac{1}{u}\text{ to }-\frac{1}{v}\end{array}\right\}$ by changing the signs of $\left\{\begin{array}{l}x\\ \alpha\end{array}\right.$;

$$\therefore\ -\left(\frac{1}{s}-\frac{1}{v}\right)^2\left(\frac{1}{s}-\frac{\mu+1}{v}\right)$$

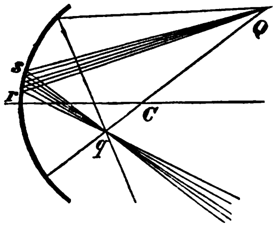
$$=-\frac{1}{8f^3(\mu-1)^3}\{x-(\mu-1)\,\alpha-\mu\}^2\{x-(\mu^2-1)\,\alpha-\mu^2\}\;\ldots\text{(ii)}.$$

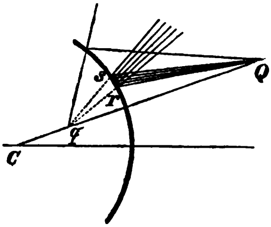
In adding, (i) and (ii) terms of odd dimension in x and α disappear, and the sum becomes

$$\frac{\mu}{4(\mu-1)^2f^3}\left\{\frac{\mu+2}{\mu-1}\,x^2-4(\mu+1)\,\alpha x+(\mu-1)\,(3\mu+2)\,\alpha^2+\frac{\mu^3}{\mu-1}\right\},$$

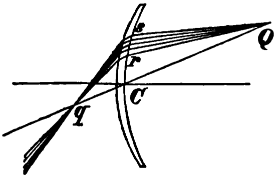
is to be a *minimum* by the variation of x;

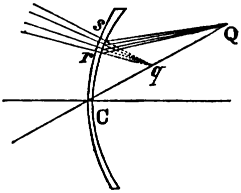
ay generally—in the cases which occur in Optical Instruents where the obliquity is small—regard,





(i) the reflected pencil as converging to or diverging rom a point q,—Q, q being conjugate foci on the line QCq hich passes through C the centre of the spherical surface;

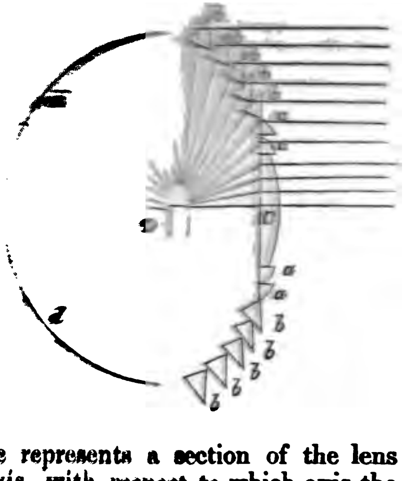




(ii) the refracted pencil as diverging from or converging o a point q which lies on the line QCq passing through C the entre of the lens,—the point q being taken to be the position f the circle of least confusion—or else determined from the quation $\frac{1}{Cq} - \frac{1}{CQ} = \frac{1}{f}$, which will generally be sufficiently approximate.

When only a small part of such a pencil exists—as the small excentrical pencil Qrs—we may still regard this excentrical portion as diverging from or converging to the same point Q after reflexion or refraction.

The student will bear in mind that this supposition is only approximately true; but it is sufficiently so for the purpose f description and explanation of pencils passing through ptical Instruments, and we shall make frequent use of it ereafter.



The figure represents a section of the lens by through its axis, with respect to which axis the lens metrical.

The central part *C* is the same as the section of lens, the light *O* being at its principal focus,—so that at emergence from *C* are parallel to *OC*. *a*, *a* ... are of prisms,—the front surfaces of which are however so convex,—so that emerge from them parallel

3 ... another set of prisms [illegible] after internal reflection [illegible] spherical reflector whose centre is O.

(2) If we suppose the figure to revolve [illegible] there will be formed an annular lens [illegible] and the [illegible] transmitted through the prisms [illegible] lenses surrounding the central [illegible] the lens C. In this [illegible] are omitted, and when [illegible] fixed in a polygonal frame which [illegible] vertical axis passing through O, [illegible] transmitted through these annular lenses [illegible] produce a *revolving* or *periodic light.*

(3) If we suppose the figure to revolve through any angle about a vertical axis through O there will be formed a cylindrical lens, and the light transmitted through the several parts of it will form a horizontal beam in the form of a sector of a circle embracing the part of the horizon within which it is intended to be seen, and constituting a fixed or *permanent* light.

The rays diverging from O in the direction opposite to that of the lens are returned by means of the reflector [illegible], and from the fact that by means of the lens prisms and reflector, the rays which diverge from O in every direction are rendered serviceable, this arrangement is called *holophotal.*

The light O consists in general of a set of Argand lamps.

For further information on this subject the student may consult the article *Lighthouses*, in the *Encyclopædia Britannica*.

CHAPTER VI.

REFRACTION THROUGH MEDIA OF VARYING DENSITY:—REFLEXION AND REFRACTION AT A SURFACE IN ANY MANNER.

118. WHEN a ray traverses a medium the density of which varies continuously from point to point, we may regard the value of μ,—the absolute refractive index,—at any point as a function of the position of the point; and the equation $\mu = C$, a constant quantity, would belong to a surface at every point of which the absolute refractive index is the same,—and the form of this surface will indicate the law of stratification of the medium. By varying this constant continuously we should obtain the consecutive surfaces of equal refractive index in the medium; and if μ, $\mu + d\mu$ be the *absolute* indices of refraction for two consecutive strata, the *relative* index from the former of the two into the latter will be $\dfrac{\mu + d\mu}{\mu}$. (Art. 83, 84.)

When the change of density and therefore of refractive index is continuous, the ordinary law of refraction will lead to a differential equation, the solution of which will enable us to determine the path of the ray.

119. Suppose a ray passing from a stratum A into another B,—μ, $\mu + d\mu$ the absolute refractive indices of A and B,—ϕ, $\phi + d\phi$ the angles of incidence and refraction at the common surface of A, B:—then we have (Art. 10),

$$\sin \phi = \frac{\mu + d\mu}{\mu} . \sin (\phi + d\phi),$$

or neglecting squares and products of small quantities,

$$0 = \frac{d\mu}{\mu} \sin \phi + d\phi \cos \phi ;$$

$$\therefore\ d\phi = -\frac{d\mu}{\mu} \tan \phi ;$$

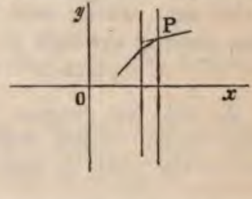
. e. when the difference of refractive index is small, the *deviation* varies as *the tangent of the angle of incidence.*

Obs. The coefficient $-\frac{d\mu}{\mu}$ in accordance with the results of experiment is assumed to be proportional to the difference of density of the strata, when that difference is small.

120. The following examples will illustrate the application of the preceding principles.

To find the path of a ray in a medium, the density at any point of which varies as its distance from a fixed plane.

The ray will pass in one plane—which take as plane of xy; and the axis of x perpendicular to the fixed plane.



λx the density at a distance x from the fixed plane.

Then (Art. 119, *Obs.*)

$$\frac{d\mu}{\mu} \propto \lambda \,.\, dx = \kappa \,.\, dx, \text{ suppose,}$$

$$\therefore \mu = A \,.\, \epsilon^{\kappa x},$$

when $x = 0$, $\mu = 1$; $\therefore A = 1$, and $\mu = \epsilon^{\kappa x}$.

Again, at any point, if ϕ be the angle which the tangent to the path of the ray makes with axis of x,—which angle is the angle of incidence,—

$$\sin\phi = \frac{\mu + d\mu}{\mu} \,.\, \sin(\phi + d\phi),$$

whence $$d\phi \,.\, \cot\phi = -\frac{d\mu}{\mu} = -\kappa \,.\, dx;$$

$$\therefore \log \sin\phi = \log C - \kappa x;$$

$$\therefore \sin\phi = \sin\alpha \,.\, \epsilon^{-\kappa x}, \qquad \text{if } \phi = \alpha \text{ when } x = 0.$$

And $$\frac{dy}{dx} = \tan\phi;$$

$$\therefore \frac{dy}{dx} = \frac{\sin\alpha \,.\, \epsilon^{-\kappa x}}{\sqrt{(1 - \sin^2\alpha \,.\, \epsilon^{-2\kappa x})}};$$

[illegible]

any two of which [illegible]
the ray.

125. Suppo[illegible] [illegible]

ay appreciably be regarded as coincident. This will
tly the case at the centre of the field of view of a tele-
-and when the field of view is not large it may be
d as true over its whole extent.

hen we speak of *image* simply, we shall in general mean
metrical image.

. DEF. If an image consist of points through which
ht actually passes it is called *real*;—in other cases *vir-*

nce a screen placed in the position of an image will
illumination only when the image is real.

amiliar instance of a *virtual* image is that formed by
non looking-glass of an object in front of it:—the
of an object under water is *virtual*. The images formed
object glass of an astronomical telescope—by the large
of a Gregorian Telescope—by a camera obscura—are

rther, a real visible object and its optical image differ
respect—from the former, light emanates in every di-
, and it can be seen in any direction, if nothing opaque
posed between it and the eye,—an image can only be
hen the eye is placed in the pencil of rays which go to
, or diverge from it. If however the image be received
creen,—such as paper or roughened glass—the light
uting the image illuminates the screen like a picture,
can be viewed by the eye as if it were a real object.

. DEF. An image is *erect* when corresponding points
bject and image are on the same side of the axis of the
ng or refracting surface—*inverted*, when they are on
e sides.

the former case the corresponding parts of the object
age appear in the same relative directions in space—in
ter, the image is inverted as regards *up* and *down*, and
versed as regards *right* and *left*. Considering the line
t along the axis of the lens or mirror as the centre of
d of view, if the image were turned through 180° about
is, the parts of the image would then be in the same
position as the corresponding parts of the object.

ent—meet the sphe
intersection of the tw

n at 1st surface,

.... 2nd surface,

planes of *firs*
plane, *i.e.* the

of *first* and

the *second,*

the relations

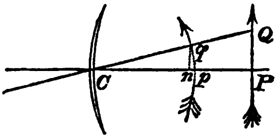
ψ) cos ω }.

the circum-
quantities

same, the
and $\mu . \mu' = 1$.
we can show

Obs. In a similar way the geometrical image of a straight ne formed by reflexion at a spherical mirror may be examined, d will be found to be a conic section.

145. Cor. 1. It may be useful to examine the more curate value of the curvature at the vertex of the image, hen the primary focus of an emergent pencil is taken to be e image of the corresponding point.



Then $Cq = v_1$, $CQ = u = \dfrac{a}{\cos\phi}$ connected by the formula

$$\frac{1}{v_1} - \frac{1}{u} = \frac{\mu\cos\phi' - \cos\phi}{\cos^2\phi\,.\,(\mu-1)f} \quad \text{(Art. 112)};$$

$$\therefore\ \frac{1}{v_1} = \frac{\cos\phi}{a} + \frac{\mu\cos\phi' - \cos\phi}{\cos^2\phi\,(\mu-1)f}.$$

Let $Cp = V$, then $\dfrac{1}{V} = \dfrac{1}{a} + \dfrac{1}{f}$,

also $Cn = v_1\cos\phi$, $qn = v_1\sin\phi$;

$$\therefore\ \rho = \text{rad. curv. at } p = \frac{1}{2}\,\text{Lt.}\,\frac{(qn)^2}{pn} = \frac{1}{2}\,\text{Lt.}\,\frac{v_1{}^2\sin^2\phi}{V - v_1\cos\phi}$$

$$= \frac{1}{2}\,\text{Lt.}\,\frac{v_1\sin^2\phi}{V\left\{\dfrac{\cos\phi}{a} + \dfrac{\mu\cos\phi' - \cos\phi}{\cos^2\phi\,(\mu-1)f}\right\} - \cos\phi}$$

$$= \frac{1}{2}\,\text{Lt.}\,\frac{v_1}{V}\cdot\frac{\sin^2\phi}{\dfrac{\cos\phi}{a} + \dfrac{\mu\cos\phi' - \cos\phi}{\cos^2\phi\,(\mu-1)f} - \cos\phi\left(\dfrac{1}{a} + \dfrac{1}{f}\right)}.$$

Now $\phi' = \dfrac{\phi}{\mu}$, nearly;—and Lt. $\dfrac{v_1}{V} = 1$ when $\phi = 0$;

telescope, since a *plane* surface in front of the object-glass would have for its image—at the principal focus of the object-glass—a surface concave towards the object-glass, and therefore *convex* towards the eye-glass; on which convex surface the objects contained in the original plane field observed, would appear to be distributed.

Again, let the points of the image and object be referred to polar co-ordinates (r, θ)—the radius vector r being measured perpendicular to the axis of the lens or mirror, and the $\angle\theta$ being the angle which r makes with a fixed plane passing through the same axis. Then if the ratio of the radii vectores of any point of the image and of the corresponding point of the object is not constant when different points are taken—there is *linear distortion* of the image.

And when the values of θ are not the same for pairs of corresponding points of the image and object,—there is *angular distortion.*

When an image is formed by *centrical* pencils, neither linear nor angular distortion will have any existence. When the image is formed by *excentrical* pencils—as in the case of the eye-glass of a telescope—there will be linear distortion in the image:—and if the eye be not on the axis of the lens, or if the surfaces of the lens be not surfaces of revolution, there will be angular distortion also.

Obs. In all cases of oblique reflexion or refraction, the image does not in general consist of an assemblage of *points*, but of circles of confusion overlapping one another, and it is therefore called *indistinct.*

This cause of imperfection of an image cannot be altogether obviated, and it will in general vary at different points of the image:—the magnitude of the circles of confusion at different points might be taken as a measure of the comparative indistinctness.

148. *Caustics.*

In Chapter III. we have indicated the general method which must be adopted to find the caustic formed by reflexion or refraction in any case, and we have there given some examples of their formation and of their use in enabling us to construct the visible image of an object. We will here give a few general theorems respecting caustics and a general explanation of the apparent deformation of an object seen by reflexion at a mirror,—or through a lens.

149. (i) Suppose an object is viewed through a convex lens. Let $QRSE$ be the course of the axis of a pencil by which any point of it is seen. Then this pencil is deflected by the lens in the same way as it would be by passing through a prism whose faces coincide with the tangent planes to the surfaces of the lens at R and S. The edge of such a prism would be turned *from* the axis of the lens and the deviation of the pencil would be greater the greater the angle of the prism,—so that the angular displacement of points of the image from the axis of the lens would be greater than in proportion to the distance of the corresponding points of the object from the same axis. Hence the distortion of an object like fig. 1 would in character resemble that given in fig. 2.

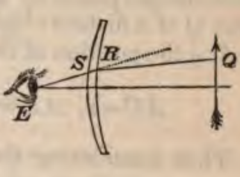


Fig 1.

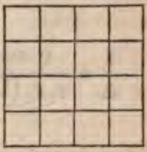
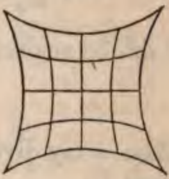
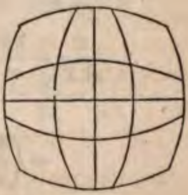


Fig. 2.



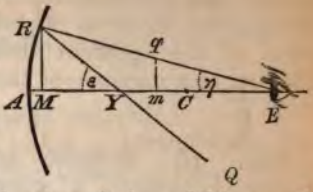
(ii) If an object be viewed through a concave lens, a similar mode of reasoning will shew that the distorted image of fig. 1 would resemble the appearance of fig. 3.

Fig. 3.



This accounts for the distortion of objects seen through spectacles.

150. *To estimate the distortion of a distant object viewed by reflexion at a concave or convex mirror.*



Let E be the position of the eye, C the centre of the surface, so that EC is the axis of visual reference.

Let QRE be the axis of the small oblique pencil by which any point Q of a distant object is seen by E, q the position of the circle of least confusion of this pencil,

$$AC = r,\ AE = x,\ AR = y;\ \therefore\ AM = \frac{y^2}{2r}.$$

Then considering the rays from Q to be parallel, we have

$$v_1 = \frac{r\cos\phi}{2},\ v_2 = \frac{r}{2\cos\phi},\ \text{and } Rq = \Delta, \text{ where}$$

$$\frac{1}{\Delta} = \frac{v_1\cos\phi + v_2}{v_1 v_2(1+\cos\phi)} = \frac{\dfrac{\cos\phi}{v_2} + \dfrac{1}{v_1}}{1+\cos\phi} \qquad \text{(Art. 70)},$$

$$= \frac{2}{r}\left\{\frac{\cos^2\phi + \sec\phi}{1+\cos\phi}\right\} = \frac{2}{r}\left(\frac{1-\phi^2+1+\dfrac{\phi^2}{2}}{1+1-\dfrac{\phi^2}{2}}\right) = \frac{2}{r},$$

if ϕ^4 &c. be neglected;

$$\therefore\ \Delta = Rq = \frac{r}{2}.$$

Also a ray diverging from Y would after reflexion pass through E;

$$\therefore\ \frac{1}{AY} + \frac{1}{x} = \frac{2}{r} + \left(\frac{1}{r} - \frac{1}{x}\right)^2 \frac{y^2}{r} \text{ (Art. 52)};$$

$$\therefore\ \frac{x}{AY} = \frac{2x}{r} - 1 + \left(\frac{1}{r} - \frac{1}{x}\right)^2 \frac{y^2 x}{r}$$

$$= \left(\frac{2x}{r} - 1\right)\left\{1 + \frac{\left(\dfrac{1}{r} - \dfrac{1}{x}\right)^2}{\dfrac{2}{r} - \dfrac{1}{x}} \cdot \frac{y^2}{r}\right\} \text{............(i)};$$

Now the amount of illumination emanating from the object and intercepted by the lens may be measured by

$$\text{area object} \,.\, I \,.\, \frac{\pi\Delta^2}{\pi d^2} = \text{area object} \,.\, I \,.\, \frac{\Delta^2}{d^2},$$

and this is diffused over the image;

$\therefore$ illumination at a point of the image

$$= \frac{I \,.\, \Delta^2}{d^2} \,.\, \frac{\text{area of object}}{\text{area of image}}, \quad \text{but } \frac{\text{area of object}}{\text{area of image}} = \frac{d^2}{d'^2};$$

$\therefore$ illumination at a point of the image $= I \,.\, \left(\dfrac{\Delta}{d'}\right)^2$, which $\propto$ apparent magnitude of lens as seen from the image.

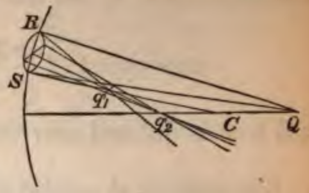
The same result will follow for the brightness of an image formed by reflexion at a spherical mirror, and is, as we see, independent of the distance of the object.

Thus the illumination of an image formed by a lens or mirror,—(supposing no light lost by reflexion or refraction or absorption by the screen which receives the image)—is the same as would be produced by the direct light of the lens or mirror if it were equally luminous with the surface of the object which emits the light.

This supposes the object to be of sensible magnitude. When the object and its image are physical points—as a star and its image—the eye judges only of absolute light, and the brightness of the image is proportional to the density of rays concentrated in it, i. e. the brightness and apparent magnitude of the lens as seen from the object. Thus in the case of a star whose distance is constant, the absolute brightness of the image is proportional to the area of the object-glass: hence the importance of large telescopes in observing very faint stars.

152. *To measure the comparative density of the rays at different points of a catacaustic.*

Let Q be the radiant point, QC the axis of the mirror, QR the axis of a small pencil QRS, of which q_1, q_2 are the primary and secondary foci after reflexion, ϕ the angle of incidence, I the intrinsic intensity of rays diverging from Q.



Then in primary plane condensation of rays at q_1

$$\propto \frac{Rq_1}{RQ},$$

and in secondary plane condensation at q_1

$$\propto \frac{1}{QR} \cdot \frac{Rq_2}{q_1 q_2};$$

therefore on the whole, condensation at q_1

$$\propto \frac{Rq_1 \cdot Rq_2}{QR^2 \cdot q_1 q_2}.$$

The above neglects the rays which may pass through q_1 after reflexion at other points of the mirror, which however would vanish compared with the condensation of rays in the primary plane.

The expression becomes indefinitely large at the geometrical focus, since $q_1 q_2$ is there $= 0$—that is, the condensation of rays at the geometrical focus is very much greater than at other points of the caustic.

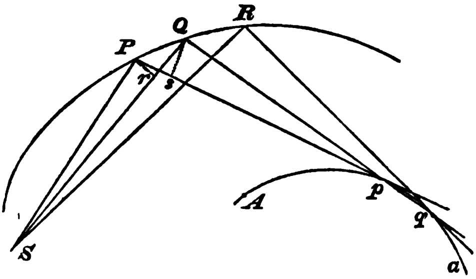
A similar measure might be obtained in the case of a diacaustic,—but such measures cannot be expressed very simply in analytical terms.

153. *Length of the curve of a caustic by reflexion or refraction.*

(i) By reflexion.

Let S be the radiant point, SP, SQ, SR three consecutive rays, the first two intersecting in p after reflexion, the last two intersecting in q. p, q are ultimately points on the caustic, and pq is ultimately a small arc of the caustic.

Let $SP=\rho$, $Pp=\rho'$, $Ap=\sigma=$ length of caustic measured



from some fixed point A on it. Draw Pr, Qs, perpendiculars from P on SQ and from Q on Pp.

Then if ϕ be the angle of incidence at P, we have

$$Qr=PQ\sin\phi=Ps, \text{ nearly,}$$

and therefore if $\Delta\rho$ be the small change of ρ in passing from P to Q,

$$\begin{aligned}\Delta\rho=Qr=Ps=Pp-Qp&=Pp-(Qq-pq)\\&=\rho'-(\rho'+\Delta\rho')+\Delta\sigma;\end{aligned}$$

$$\therefore\ \Delta\sigma=\Delta\rho+\Delta\rho'.$$

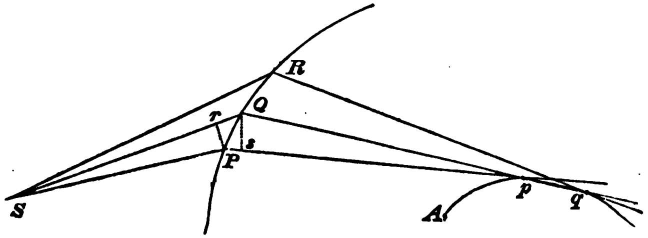
This which is approximately true becomes strictly true in the limit; therefore integrating

$$\sigma=\rho+\rho'+\text{a constant.}$$

ρ, ρ' are supposed to be measured positive in the direction in which light is propagated.

(ii) By refraction.

With notation similar to the preceding,



μ being the index of refraction from first medium second, we have

$$\Delta\rho = Qr = PQ\,.\sin\phi = \mu PQ\sin\phi'$$

$$= \mu Ps = \mu\,(Pp - Qp)$$

$$= \mu\,\{Pp - (Qq - pq)\}$$

$$= \mu\,\{\rho' - (\rho' + \Delta\rho') + \Delta\sigma\}\,;$$

$$\therefore\ \Delta\sigma = \frac{\Delta\rho}{\mu} + \Delta\rho'\,;$$

$$\therefore\ \sigma = \frac{\rho}{\mu} + \rho' + \text{a constant.}$$

Thus we see that the length of any catacau- caustic corresponding to any pencil in one plan pressed in algebraic terms.

In each of the above cases (i), (ii), ρ, ρ' are be measured positive in the direction in which pagated.

Analytical investigations of the above res in Herschel's *Light* and other works.

Cor. When two consecutive reflected or are parallel, it indicates an asymptote in the ca any variable which defines the direction of a ψ the angle which the corresponding reflect ray makes with a fixed direction,—then for a the caustic we must have $\dfrac{d\psi}{d\phi} = 0$.

If the value of $\dfrac{d\psi}{d\phi}$ does not vanish for value of ϕ, there is no asymptote.

154. *The form of a caustic near a cusp parabola.*

larly on a screen,
n. On turning the
ctrum will descend
nd. It will be in
a position of mini-
pectrum remaining
f the prism about
n be examined in
to CB, it is found
he circular spot B,
nt colours in diffe-
or least refracted end,
ing *orange, yellow,*
s we proceed to the

asses with minimum
sm, and very near its
refracted pencil would
a distance $= CA$ from
ameter for its vertical
being received perpen-
$D = CB$ the appearance
l to B. The experiment
eory:

t species of light of all
certain limits, and of all
s are the least, and the violet

the spectrum in this experiment,
hich was observed by Newton with
ts amount in different cases depends
he prism and its material.

n formed in the manner just described
is, the coloured light at any point of it
ure of rays of a higher degree of refran-
point of the sun's disc with rays of a lower
bility from a lower point; and consequently
the rays of different degrees of refrangibility

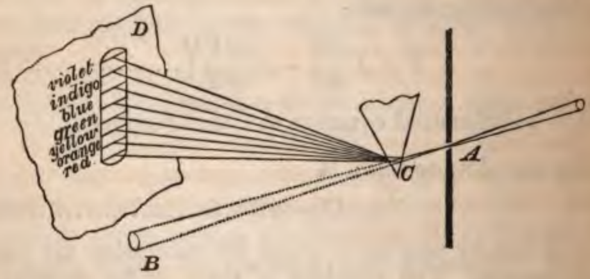
CHAPTER IX.

ON THE CHROMATIC DISPERSION OF LIGHT.

155. In the previous chapters pencils of light have been considered homogeneous. The following experiment of Newton shews that a pencil of sunlight has not this uniform nature, but admits of decomposition into a system of pencils, in each of which the rays have a peculiar degree of refrangibility.

156. *Newton's Experiment.*

If the light of the sun be admitted into a darkened room through a small aperture *A* in a window-shutter, the pencil of light after entering the room may be regarded as approximately a cone, with *A* for its vertex, and the sun's apparent diameter for its vertical angle. If this pencil be allowed to fall perpendicularly on a screen, a circular bright spot *B* of white light is visible. Let the pencil now be refracted upwards through a prism of glass, *or any other refracting substance*, placed very near to the aperture *A*, the axis of the pencil passing perpendicularly to the edge of the prism *C*, which for convenience of reference we will suppose horizontal, and very near to it.



If the pencil be now received perpendicularly on a screen, an elongated stripe, *or spectrum* D is seen. On turning the prism continuously about its edge, this spectrum will descend to a certain limiting position, and then ascend. It will be in this limiting position when the prism is in a position of minimum deviation, which is indicated by the spectrum remaining stationary for a *very small* angular motion of the prism about its edge in *either* direction. If the spectrum be examined in this position, the distance CD being equal to CB, it is found to be of the same horizontal breadth as the circular spot B, but about five times longer, and of different colours in different parts, being *red* at the lowest, or least refracted end, then by a gradual change of tint becoming *orange*, *yellow*, *green*, *blue*, *indigo*, *violet* in succession, as we proceed to the upper extremity.

Now, since the axis of the pencil passes with minimum deviation in a principal plane of the prism, and very near its edge,—if light were homogeneous the refracted pencil would be a cone diverging from an origin at a distance $= CA$ from C, and having the sun's apparent diameter for its vertical angle (Art. 96, Cor. 2). This cone being received perpendicularly on a screen at a distance $CD = CB$ the appearance would be a circular spot exactly equal to B. The experiment therefore leads to the following theory:

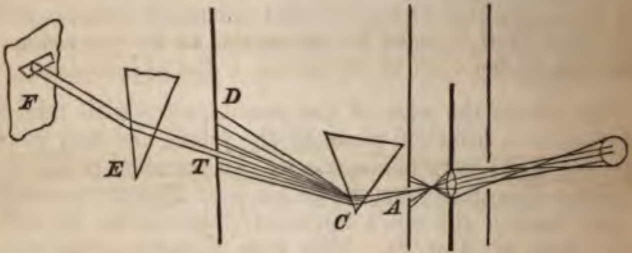
Sun-light consists of different species of light of all degrees of refrangibility within certain limits, and of all varieties of colour. The red rays are the least, and the violet rays the most refrangible.

Obs. The elongation of the spectrum in this experiment, as stated above, is that which was observed by Newton with the prism used by him. Its amount in different cases depends on the refracting angle of the prism and its material.

157. The spectrum formed in the manner just described will not be *pure*, that is, the coloured light at any point of it will consist of a mixture of rays of a higher degree of refrangibility from one point of the sun's disc with rays of a lower degree of refrangibility from a lower point; and consequently in this spectrum the rays of different degrees of refrangibility

are not accurately separated. The defect arising from this over-lapping of the spectra which correspond to pencils emanating from successive points of the sun's disc, may be diminished by receiving the pencil which enters at *A* (fig. p. 142) on a screen perforated so as to allow only a small part of this pencil to pass, and thus the pencil incident upon the prism would have a smaller vertical angle, and diverge more nearly from a point.

But a better plan is to obtain a *very small* image of the sun by means of a lens of small focal length (as in the figure),



and a small part of the pencil diverging from this image may be allowed to pass through the aperture *A* and received on the prism.

If, for example, the focal length of the lens be 1 inch, the diameter of the sun's image formed at its principal focus = tan (sun's apparent diameter) = tan 30′ nearly = $\frac{1}{114}$ inch, which is so small a quantity that for this purpose it may be regarded as a physical point.

If the screen *D* be moveable, and perforated at *T*, any part of the spectrum first formed can be *insulated* and separately examined *or analyzed* by a second prism *E*.

By this means a high degree of purity in the spectrum may be obtained; when this is done, it is found that the colours admit of no further decomposition, since when analyzed by a second prism at *E*, and received on a screen *F*, the appearance there is a spot of uniform colour, the same as at *T*, and of dimension the same as if the pencil at *T* had been allowed to pass uninterrupted to the screen *F*.

further from *G*, a similar band is seen. From *H* to *I*, the end of the spectrum, the lines are equally numerous.

Two of the fixed lines, probably *E* and *F*, had been discovered by Wollaston previously to the experiments of Fraunhofer—but he did not pursue the investigation of them.

161. The name of *fixed lines* was given by Fraunhofer to these dark bands which interrupt the continuity of the colours. As long as the source of light remains the same, these lines occur in the *same order* and in the *same colours*, whatever be the substance of which the prism is formed,—their relative distances only varying when different substances are employed.

Thus for instance with solar light—whether obtained directly from the Sun, or indirectly, as from the clouds or sky, from the Moon or planets, or from a rainbow—the phenomena of the dark lines in the spectrum, as to order, number, and relative position, is always the same when the same substance is used.

When the Sun is very near the horizon the blue end of the spectrum disappears, and lines are seen in the remaining part which were not before visible.

Analogous effects are produced by interposing certain coloured vapours and liquids between the slit *A* and the source of light.

In the spectrum given by the light of various fixed stars lines are observed differently situated from those of the solar spectrum. When the flame of a candle or oil-lamp is placed behind *A* the spectrum is not interrupted by dark lines:—a bright double line however is seen exactly occupying the place of the double line *D*.

When the flame of a spirit-lamp is placed behind *A*, the double line *D* is extremely bright compared with the rest of the spectrum. The spectrum formed by electrical light consists almost entirely of a few bright lines, some of which, according to the observations of Prof. Wheatstone, appear to depend upon the nature of the substances between which the spark is produced.

162. Fraunhofer selected the fixed lines *B*, *C*, *D*, *E*, *F*, *G*, *H* as points of reference in the spectrum, and by measuring (Art. 165) with extreme accuracy the deviations of these lines through prisms of the same substance with different refracting angles, he ascertained that for a ray corresponding to any one of the fixed lines, the ratio of the sine of the angle of incidence to the sine of the angle of refraction was invariable,—thus affording the strongest corroboration of the law of refraction (Art. 9).

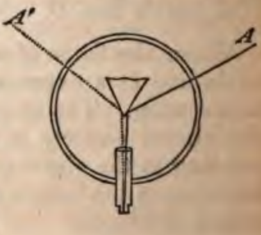
The following table contains some of the results of Fraunhofer's observations as given in his *Essay on the Determination of Refractive Powers, &c.*—the refractive index of a fixed line being understood to mean that of a ray which would suffer the same deviation, if it existed.

	Sp. G.	μ_B	μ_C	μ_D	μ_E	μ_F	μ_G	μ_H
Water at 15° *R*	1·000	1·33095	1·33171	1·33357	1·33585	1·33780	1·34127	1·34417
Crown Glass, No. 9	2·535	1·52583	1·52685	1·52959	1·53301	1·53605	1·54166	1·54657
Flint Glass, No. 13	3·723	1·62775	1·62968	1·63504	1·64202	1·64826	1·66029	1·67106
Oil of Turpentine	0·885	1·47050	1·47153	1·47443	1·47835	1·48174	1·48820	1·49387

163. The following two propositions will indicate the nature of the observations requisite for determining the index of refraction of a ray of any colour—the ray being defined by its position relative to the fixed lines of the spectrum.

164. *To measure the refracting angle of a prism.*

Let the prism be firmly fixed to a graduated circle provided with verniers,—the edge of the prism being at the centre of the circle, perpendicular to its plane, and turned towards the object-glass of a telescope fixed to the circle. Let the circle be turned until the image of a well-defined distant object *A*, seen by reflexion at one of the *faces* of the prism, and viewed

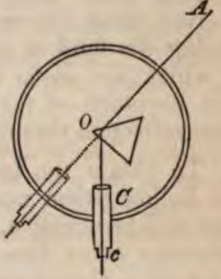


through the telescope, coincides with the intersection of the cross-wires at the principal focus of the object-glass of the telescope—and read off the verniers attached to the circle.

Turn the circle about its centre till the image of the same object seen by reflexion at the other face of the prism coincides with the intersection of the cross-wires—the effect will be the same as if we supposed the circle to be stationary, and the object A to revolve about the centre of the circle into a position A' such that it is seen by reflexion at the second face of the prism, and read off the verniers again. The *difference of the readings*—or the angle through which the circle has been turned—is equal to twice the angle of the prism.

165. *To measure the minimum deviation of a ray corresponding to one of the fixed lines, out of air into any medium formed into a prism; and thence to determine its index of refraction.*

Let O be the centre of a graduated circle moveable round an axis perpendicular to its own plane on a fixed circle carrying verniers, Cc a telescope the axis of the object-glass of which passes through the axis of the circle and is parallel to the plane of the circle,—A the intersection of the plane described by the axis of the telescope with the slit perpendicular to the plane of the circle, by which light is admitted. The telescope is provided with cross-wires at such a distance from the object-glass that when it is pointed to A, the image of A may be at the intersection of the wires. Place the prism with its edge perpendicular to the plane of the circle, so that its faces may be equidistant from the axis of the circle. Turn the prism until the incident ray and the given emergent ray make equal angles with the faces of the prism—*i.e.* until the deviation of this particular ray is a minimum, or the image of the fixed line stationary. The prism retaining this position, turn the telescope till the image of the line coincides with the intersection of the wires and read the verniers.



Now remove the prism and turn the telescope until the intersection of the wires coincides with the image of A and read the verniers again. The difference of the two readings—or the angle through which the circle has been turned between the two observations—is the *minimum deviation* of the given ray.

If D be this deviation, ι the refracting angle of the prism, μ the index of refraction for the given colour, or fixed line, then with the notation of Art. (95),

$$D = 2\phi - \iota, \quad \iota = 2\phi';$$

$$\therefore \mu = \frac{\sin\phi}{\sin\phi'} = \frac{\sin\dfrac{D+\iota}{2}}{\sin\dfrac{\iota}{2}},$$

from which μ can be calculated numerically.

166. *Obs.* The refractive index of a fluid or a gas for a given fixed line may be found in the same manner by enclosing the fluid or gas in a hollow prism of glass, the sides of which are *plates* with their surfaces accurately parallel. The deviation of the axis of the pencil then arises entirely from refraction through the fluid.

To obtain the index of refraction from air into *vacuum*, the hollow prism must be *exhausted:* the deviation in this case will be negative, and the value of $\mu < 1$. Of course the refractive index from vacuum into air is the reciprocal of the value of μ thus found: the values of the *absolute* refractive indices can then be calculated (Art. 84).

Note. When the prism is not sufficiently perfect to exhibit the fixed lines, we must select by estimation the particular part of the spectrum for which the index of refraction is required,—and measure the deviation as above.

As the spectrum produced by a given species of light—sun-light for instance—refracted through a given substance always presents the same phenomena, we may regard it as a standard scale of colours which can be reproduced at any time, and in which any particular tint may be defined and identified by reference to the fixed lines.

Thus the value of μ for any particular medium or substance is not invariable, but is susceptible of every possible value between certain limits: but rays corresponding to each particular value of μ will conform to the laws of reflexion and refraction which have been discussed in the previous chapters on the supposition of light being homogeneous.

167. *Def.* A ray of white light being decomposed by refraction at any surface into a beam of coloured rays, the angle between any coloured ray, or ray corresponding to a fixed line, and the original white ray produced, is the *deviation* of that colour or fixed line.

The difference of the deviations of two colours, or fixed lines, is the *dispersion* of those colours or fixed lines.

A discussion of the values of μ obtained in Fraunhofer's manner leads to the conclusion,—that the *ratio* of the dispersion of any two colours to the dispersion of the extreme colours of the spectrum, is not constant when different media are used. This fact is called the *Irrationality of dispersion.*

Hence if two prisms be formed of different substances and of such refracting angles as to give spectra of the same total length, i.e. equal total dispersion,—and they be placed so as to refract a pencil of white light in opposite directions,—in the emergent beam the extreme red and violet will be united but the intermediate colours will not be completely so, but will give rise to a spectrum of faint colours and small breadth. Such spectra which exist in consequence of the irrationality of dispersion are called *secondary spectra.*

168. *When a ray of white light is refracted through a prism in a principal plane, to find the dispersion of two colours of given refractive index.*

Let ι be the refracting angle of the prism, ϕ the angle of incidence of the white ray, ϕ' the angle of refraction of the coloured ray for which μ is the refractive index,—and ψ', ψ

angles of incidence and emergence of the same at the second surface,—D the deviation of this colour. Then

$$\sin\phi = \mu\sin\phi', \qquad \sin\psi = \mu\sin\psi';$$

$$D = \phi + \psi - \iota, \qquad \iota = \phi' + \psi' \quad \text{(Art. 92. } Obs. \text{ 2).}$$

If $\mu + \delta\mu$ be the index of refraction for another of the colours into which the incident ray is separated by refraction, $\delta\mu$ being small—$D + \delta D$ the deviation of this second colour—then will δD be the dispersion required.

Treating μ as an independent variable, since it is susceptible of all values between the limits corresponding to the extreme colours, ϕ being constant,

$$\delta D = \frac{d\psi}{d\mu}\,.\,\delta\mu, \qquad 0 = \frac{d\phi'}{d\mu} + \frac{d\psi'}{d\mu},$$

$$\cos\psi\,\frac{d\psi}{d\mu} = \sin\psi' + \mu\cos\psi'\,\frac{d\psi'}{d\mu},$$

$$0 = \sin\phi' + \mu\cos\phi'\,\frac{d\phi}{d\mu};$$

$$\therefore\ \frac{\cos\psi}{\cos\psi'}\,.\,\frac{d\psi}{d\mu} = \frac{\sin\psi'}{\cos\psi'} + \frac{\sin\phi'}{\cos\phi'} = \frac{\sin\iota}{\cos\phi'\cos\psi'};$$

therefore the dispersion

$$= \delta D = \frac{d\psi}{d\mu}\,.\,\delta\mu = \frac{\sin\iota}{\cos\psi\cos\phi'}\,.\,\delta\mu.$$

169. Cor. The above expression for the dispersion cannot vanish for any position of the prism. It admits however of a minimum value, which will happen when

$$\cos\psi\cos\phi' = \textit{maximum}.$$

When such is the case, μ being constant in this problem,

$$\tan\psi\,.\,d\psi + \tan\phi'\,.\,d\phi' = 0;$$

$$\therefore \frac{\sin \iota}{\cos \phi' \cos \psi} d\mu = \frac{\sin \iota_1}{\cos \phi_1' \cos \psi_1} d\mu_1,$$

the condition required.

If the prisms are of the same substance, then $d\mu = d\mu_1$.

176. *Note.* Since only a limited number of colours can be united by an achromatic combination,—it will be best to select colours for this purpose which have a brilliant illuminating power and also differ considerably in colour—since by this course a far greater concentration of light and a more approximate union of the remaining colours will be produced, than if we attempted to unite the extreme red and violet rays which are too little luminous to render their union a matter of importance.

For example, if *two* colours only are to be united, the best to be selected for this purpose are those defined by the fixed lines C and F, or D and F (Art. 160). If there be a sufficient number of disposable quantities to enable us to unite *three* colours, the best to be selected would be the fixed lines C, E, and G,—or C, F, and a colour midway between D and E.

177. *Achromatic Lenses.*

In forming an achromatic combination of lenses the problem is different according as the refracted pencil passes *centrically* or *excentrically*. In the former case the pencils of different colours into which an incident pencil is separated have a common axis (Art. 111) in which their points of divergence or convergence lie (Art. 113), and the combination is achromatised by making as many as possible of these points coincide. But in a pencil refracted excentrically, the axes of the coloured pencils after refraction have different directions so that there is an angular separation of their points of divergence or convergence—as seen by an eye in a given position—and the condition of achromatism will be that which will render the axes of as many as possible of these coloured pencils parallel—since rays which enter the eye in state of parallelism affect the eye in the same way as if they were coincident—or very nearly so.

171. *Def.* If μ_r, μ_v, μ be the in
the extreme red and violet rays capable o
impression on the eye of the observer,
refrangibility out of air into any medium

$$\frac{\mu_v - \mu_r}{\mu - 1}$$

is called the dispersive power of the me
is frequently denoted by the letter ϖ.

If the medium be formed into a
refracting angle i, and if D_r, D_v, D be t
extreme red and violet rays and for rays o
of the axis of a pencil which passes th
principal plane and at a small angle of
gence, then

$$D_r = (\mu_r - 1)\, i, \quad (A$$

$$D_v = (\mu_v - 1)\, i,$$

$$D = (\mu - 1)\, i;$$

$$\therefore \frac{D_v - D_r}{D} = \frac{\mu_v - \mu_r}{\mu - 1};$$

or the dispersive power of a medium form
is equal to the ratio of the total *dispersio*
extreme red and violet pencils to the *devi*
the pencil of light of mean refrangibility.

For an account of dispersive powers o
order to which the irrationality of disper
student is referred to Herschel's *Treatise on*

Achromatic Combination

172. A combination of prisms or lenses i
matic when the *dispersion* of the pencils
through them is reduced within the narrow
We proceed to consider the conditions whi
in such combinations.

A pencil of sun-light after refraction does not in general converge to or diverge from a point, for two reasons; (i) from the unequal refrangibility of the different species of light of which it is composed, and (ii) in consequence of the finite breadth of the pencil and the curvature of the refracting surface. These causes of aberration being independent of one another may be separately considered. It will therefore be supposed for simplicity in the following articles, in investigating the conditions of achromatism, that no *spherical aberration*—(i. e. aberration arising from the *curvature* of the refracting surfaces,)—exists in the pencils which we consider. Moreover we shall consider it sufficient to obtain the conditions of achromatism *for the axis of a pencil*, since if these conditions are satisfied for the axis, they will be so very approximately for the other rays of the pencil,—supposing it small.

When we speak of *a colour* it must be understood to be defined by its position among the *fixed lines* of the spectrum —or to correspond itself to *a fixed line.*

173. The *possibility* of an achromatic combination arises from the fact that the *dispersion* of a ray and the *deviation* of any particular colour—or the *mean deviation*, if for convenience we take this as a definite measure of the deviation of the ray—produced by a refracting medium are not proportional. A comparison of the results given in the table (page 148) will prove this.

If *dispersion* were proportional to *deviation* for different media, then any combination which would destroy dispersion, would also destroy deviation; and in consequence would be useless for the purposes for which such combinations are designed. For instance, a combination of lenses in a telescope would have no magnifying power, if it did not produce deviation in the axis of a pencil transmitted through the telescope; but since in different media dispersion is not proportional to deviation, media can be found which produce the *same dispersion* in *opposite* directions of a given colour relatively to another,—but at the same time, *different deviations* in *opposite* directions in the axis of a pencil. If a pencil then be refracted through these media, the two colours in question can be united, while the axis of the pencil suffers a deviation equal

If this be expanded in powers of the small quantity $\frac{d\mu_1}{\mu_1 - 1}$, and powers higher than the second be neglected, this becomes

$$\frac{f_1}{(a+f_1)^2} \cdot \frac{d\mu_1}{\mu_1 - 1} - \frac{af_1}{(a+f_1)^3} \left(\frac{d\mu_1}{\mu_1 - 1}\right)^2,$$

and if μ_2 become $\mu_2 + d\mu_2$,

$$\text{the alteration of } \frac{1}{f_2} \text{ is } = \frac{d\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2}.$$

Hence if v remain the same for the two pairs of refractive indices which we have considered,—or if the combinations be achromatic for the two corresponding colours,

$$0 = \frac{f_1}{(a+f_1)^2} \cdot \frac{d\mu_1}{\mu_1 - 1} - \frac{af_1}{(a+f_1)^3} \left(\frac{d\mu_1}{\mu_1 - 1}\right)^2 + \frac{1}{f_2} \cdot \frac{d\mu_2}{\mu_2 - 1},$$

the condition of achromatism.

180. *A pencil passes excentrically through two thin lenses separated by a given interval,—its axis before incidence intersecting the common axis of the lenses in a given point:—to find the condition of achromatism.*

Let f_1, f_2 be the focal lengths of the lenses for a *fixed line* whose refractive index in them is μ_1, μ_2 respectively, a the distance between the centres of the lenses. Let the axis of the pencil of this colour before and after refraction through the first lens cut the axis of the lenses at distances b_1, c_1 from the centre of that lens,—and before and after refraction through the second lens at distances b_2, c_2 from the centre of that lens;—also let ϵ, η be its inclination to the axis of the lenses before refraction and at emergence.

Then using first approximations (Art. 137),

$$\frac{1}{c_1} - \frac{1}{b_1} = \frac{1}{f_1},$$

$$\frac{1}{c_2} - \frac{1}{b_2} = \frac{1}{f_2},$$

$$b_2 = c_1 + a,$$

$$\frac{\tan\eta}{\tan\epsilon} = \frac{b_1 b_2}{c_1 c_2};$$

$$\text{and } \therefore \frac{b_1}{c_1} = 1 + \frac{b_1}{f_1},\quad \frac{b_2}{c_2} = 1 + \frac{c_1 + a}{f_2};$$

$$\begin{aligned}\therefore \frac{\tan\eta}{\tan\epsilon} &= \frac{b_1 b_2}{c_1 c_2}\\ &= \frac{b_1}{c_1} + \frac{b_1}{f_2} + \frac{a}{f_2}\cdot\frac{b_1}{c_1}\\ &= 1 + \frac{b_1 + a}{f_2} + \frac{b_1}{f_1} + \frac{ab_1}{f_1 f_2}.\end{aligned}$$

Now if $\left.\begin{matrix}\mu_1\\ \mu_2\end{matrix}\right\}$ become $\left\{\begin{matrix}\mu_1 + d\mu_1\\ \mu_2 + d\mu_2\end{matrix}\right.$,

$$\text{the alteration of } \left.\begin{matrix}\dfrac{1}{f_1}\\ \dfrac{1}{f_2}\end{matrix}\right\} \text{ is } \left\{\begin{matrix}\dfrac{d\mu_1}{\mu_1 - 1}\cdot\dfrac{1}{f_1}\\ \dfrac{d\mu_2}{\mu_2 - 1}\cdot\dfrac{1}{f_2}\end{matrix}\right.;$$

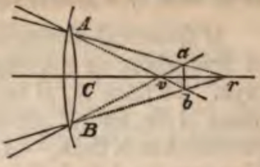
$$\begin{aligned}\text{and of } \frac{1}{f_1 f_2} \text{ is } &= \frac{1}{f_1 f_2}\left\{1 + \frac{d\mu_1}{\mu_1 - 1} + \frac{d\mu_2}{\mu_2 - 1}\right\} - \frac{1}{f_1 f_2}\\ &= \frac{1}{f_1 f_2}\left\{\frac{d\mu_1}{\mu_1 - 1} + \frac{d\mu_2}{\mu_2 - 1}\right\},\end{aligned}$$

eglecting the product of $\dfrac{d\mu_1}{\mu_1 - 1}$ and $\dfrac{d\mu_2}{\mu_2 - 1}$.

Hence, if η be the same for the two pairs of refractive ndices, or the combination be achromatic for the two correponding kinds of light,

$$0 = \frac{b_1 + a}{f_2}\cdot\frac{d\mu_2}{\mu_2 - 1} + \frac{b_1}{f_1}\cdot\frac{d\mu_1}{\mu_1 - 1} + \frac{ab_1}{f_1 f_2}\left(\frac{d\mu_1}{\mu_1 - 1} + \frac{d\mu_2}{\mu_2 - 1}\right) \dots (A),$$

he condition of achromatism.



Let C be the centre of a thin lens AB through which a encil is directly refracted, v, r the geometrical foci for the most and east refracted rays of the pencil, μ_v, μ_r the indices of refraction for he same respectively, μ the index or mean rays, v the distance from the centre of the geometrical focus of mean rays, u the distance from the centre of the origin of the pencil which we suppose to consist of white light at incidence, r, s the radii of the lens,—lines as usual being accounted positive when measured from the centre contrary to the direction of the incident pencil. Then

$$\frac{1}{Cv} - \frac{1}{u} = (\mu_v - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

$$\frac{1}{Cr} - \frac{1}{u} = (\mu_r - 1)\left(\frac{1}{r} - \frac{1}{s}\right);$$

$$\therefore\ \frac{Cr - Cv}{Cr \,.\, Cv} = (\mu_v - \mu_r)\left(\frac{1}{r} - \frac{1}{s}\right).$$

Since $Cr - Cv$ is usually small compared with Cv or Cr, we may put

$$Cr \,.\, Cv = v^2, \text{ nearly,}$$

and $$(\mu_v - \mu_r)\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{\mu_v - \mu_r}{\mu - 1} \,.\, (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{\varpi}{f},$$

if ϖ be the dispersive power of the lens (171) and f its focal length for mean rays;

$$\therefore \text{ chromatic aberration} = Cr - Cv = rv = \frac{\varpi v^2}{f}.$$

Cor. Of the two points, r, v,—v is the nearer to C, *except* when the origin of the pencil is between the centre of the lens and the principal focus of rays incident in the contrary direction.

spectrum in which it happens to be placed, but its hue is much more brilliant and vivid when placed in a part of the spectrum analogous to its own natural colour. Hence we are led to conclude that the colours of natural bodies are not qualities inherent in themselves, but arise from their aptitude to absorb and reflect different kinds of light. Thus a substance appears *green*—suppose—in consequence of its disposition to reflect green rays alone, the other prismatic colours of which the white solar light falling upon it is composed, being stifled or absorbed by it.

In forming his theory of the composition of colours, Mayer regarded all colours as resulting from the combination of *three primary colours,—red*, *yellow* and *blue*. And he gave as results of his experiments the different proportions in which these colours entered into combination to produce others. Dr Young assumed *red*, *green* and *violet* as primary colours, and states that *white* light is composed of these *complementary* colours in the proportions of two parts of red, four of green, and one of violet. Sir D. Brewster regards red, yellow and blue as primary colours, a certain portion of each existing at every point of the spectrum—the colour being determined by the predominating colour at that point, mixed with white light.

See Dr Young's *Lectures on Natural Philosophy; Trans. Roy. Soc. Edin.* Vol. XII.; Chevreul, *Cercles Chromatiques.*

is in the axis of the eye. Over the back of the eye there extends a black velvety substance, called the *pigmentum nigrum*, perfectly incapable of reflecting light—and in this is embedded a delicate tissue of nerves called the *retina*, which communicate with the brain by means of the optic nerve *K*.

B is a soft transparent jelly-like substance—called the *crystalline lens*—in the form of a double convex lens having its axis coincident with *ACD* the axis of the eye, and held in its place by tendons springing from the choroïdes. The spaces between the cornea and the crystalline, and between the crystalline and the retina, are filled with transparent fluids, called respectively the *aqueous* and the *vitreous* humours. The refractive index of these humours out of air is nearly that of water:—the refractive index of the crystalline is a little greater.

188. *To explain the manner in which vision takes place.*

Let the axis of the eye be directed to a point of an object *PQ*. A pencil from any point *P* falls upon and is refracted by the cornea. Of this pencil a portion limited by the aperture *GH* is again refracted by the aqueous humour, the crystalline lens and the vitreous humour, and is made to converge very nearly to a point *p* on the retina. The impression thus made on the retina is communicated to the brain and produces the sensation of vision of the point *P*.

189. The spot *K* where the nerves of the retina pass through the coats of the eye, is insensible to vision—and from this cause is called the *punctum cæcum*. The following simple experiment is given by Sir J. Herschel for shewing the existence of this point. "On a sheet of black paper or other dark ground place two white wafers having their centres three inches distant. Vertically above that to the *left* hold the *right* eye at 12 inches from it, and so that when looking down on it the line joining the two eyes shall be parallel to that joining the centre of the wafers. In this situation, closing the left eye and looking full with the right at the wafer perpendicularly below it, this only is seen, the other being completely invisible. But if removed ever so little from its place, either right or left, above or below, it becomes immediately visible, and

starts as it were into existence. The distances here set down may perhaps vary slightly in different eyes."

190. The front part of the choroides surrounding the pupil is called the *uvea* or *iris*, which is differently coloured in different individuals, and by an involuntary action is capable of expanding or contracting the pupil within the limits of about ·25 and ·09 inches:—so as to admit a larger or smaller pencil of light as the object viewed is less or more brilliant. This involuntary action may be observed by any one watching his own eye as he walks towards a mirror, near which is placed a bright object—as a lamp for instance. The aperture of the pupil in the human eye is always circular—but in animals of the cat kind the vertical diameter appears to remain invariable, the contraction taking place in a horizontal direction,—in the eye of the horse, on the contrary, the horizontal diameter of the pupil remains nearly constant, whilst it admits of contraction in a vertical direction.

The eye is able to adapt itself automatically to objects at different distances, so as to make pencils of different degrees of divergency converge nearly to a point on the retina.

That this focal adjustment of the eye for different distances is effected—in part at least—by a change of form of the crystalline lens,—has been shewn by Prof. Helmholtz;—who by carefully observing the images of a bright object reflected at the anterior and posterior surfaces of this lens, found that as the eye adjusted itself for different distances these images underwent a change in form and position which could only be accounted for by a change of curvature of the surfaces of the lens: and this is probably attended by a change of form in the cornea likewise.

From the accurate measures which have been made it further appears that the surface of the cornea is a prolate spheroid, and the anterior and posterior surfaces of the crystalline lens are oblate spheroids:—the density also of this lens is found to increase from the outside towards the centre, the tendency of which is to correct the aberration by shortening the focal length for rays which pass near the centre. The forms of the several surfaces, as might be expected, vary in the form with greater difficulty, according as the converted form is less familiar and probable.

193. It has sometimes been felt as a difficulty, that objects appear *erect*, although the image of them on the retina is *inverted*. *Erect* and *inverted* are simply terms implying *relative* position, and as our optical knowledge of *all* external objects is derived from the picture of them on the retina, there is the same *relative* displacement of all objects as regards their disposition in space. Thus whilst the picture of men on the retina exhibits them with their heads *downwards*, at the same time heavy bodies appear to fall *upwards*. The mind forms an estimate of all the relative parts of the picture simultaneously, and its relation to external objects is judged of by experience and habit.

As however a discussion of the physiology of vision is beyond our purpose, the student who wishes to pursue the subject may consult an interesting article on *Binocular Vision* in the *Edinburgh Review* for *October*, 1858, and the authorities there referred to.

194. It may be interesting to know the dimensions of the human eye,—they will of course vary in different individuals, but the following are given as belonging to an eye of average size for a person in middle life. The axis of the eye measured from the outer surface of the cornea to the retina = ·95 inches, and the portion of this length occupied by the cornea = ·04 of an inch, the aqueous humour = ·11, crystalline lens = ·17, vitreous humour = ·63.

Interior transverse diameter of the eye	·90	inches.
Vertical chord of the cornea	·46	...
Horizontal chord of the cornea	·49	...
Chord of the crystalline lens	·37	...
Radius of external surface of cornea	·33	...
Radius of anterior surface of crystalline	·33	...
Radius of posterior surface of crystalline	·24	...

The *centre* of the eye for optical purposes is a point nearly in the centre of the pupil, in the plane of the iris.

The angle between the axis of the eye and the line oining the centre of the punctum cæcum with the centre of he eye is about 14°, the breadth of the punctum cæcum is bout $\frac{1}{6}$ inch, subtending an angle of 5° at the centre of the ye.

By the revolution of the eye in its socket, its axis has a ange of about 55° in every direction about its mean position.

The impression produced by light on the retina continues bout $\frac{7}{60}$ *of a second* after the light has ceased to act, so that f a bright object be whirled round in a circle, the period of ts gyration being less than this, it will appear as a *continuous* bright circle. With respect to the magnitude of the *minimum visibile*, or apparent magnitude of the least visible object, it is usually stated that an object is invisible to most eyes unless it subtends an angle of at least *one minute.* This however will vary for different eyes, and the *brightness* of the object must materially influence the question, since we find that the fixed stars produce a distinct and vivid impression on the retina, although the angle which they subtend to the eye is so small as to be incapable of being measured by the best instruments.

The numerical results of this Article are taken from Lloyd's *Treatise on Light and Vision.* For more minute dimensions of the different parts of the Eye, the student may consult Kölliker's *Manual of Human Histology.*

195. *Defects of Vision.*

An eye which produces too great refraction of a pencil incident upon it brings pencils from distant points to convergence at points so far before the retina as to produce no distinct impression upon it. This defect is called *short sight.*

On the other hand, for an eye which cannot sufficiently refract a pencil, the least distance of distinct vision is greater than eight inches, pencils from points within this least distance being brought to convergence behind the retina. This is *long sight.*

196. Vision through optical contrivances depends on the fact that if a pencil diverging from a given point fall on the eye, it is immaterial whether that point be an actual source of light, or whether the rays have been made to converge to it and afterwards to diverge. An image therefore is visible in the same manner as a luminous object in the same position would be, with this limitation,—that from any point of a luminous object rays diverge in all directions, but from any point of an image rays diverge only in directions corresponding to the directions of those rays which form that point in the image.

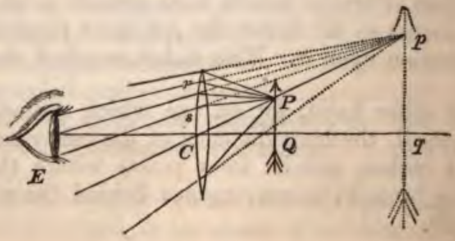
In general explanations of vision through optical instruments, *spherical aberration* may from its smallness be disregarded. Pencils may be considered after reflexion or refraction to converge to or diverge from a point, and an *excentrical* pencil may be supposed to have the same point of divergence or convergence as the centrical pencil from the same origin.

Further, when an object is at such a distance as to be conveniently seen, the pencil received by the eye from any point of it will from its smallness have a *very small* degree of divergence,—so small that for the purpose of general description of an instrument we shall regard the pencils which emerge from the eye-glass to consist of parallel rays.

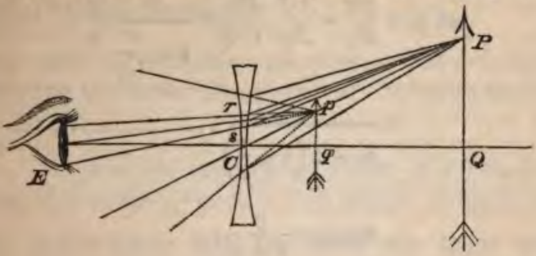
The slight adjustment requisite to give distinct vision of the image must be performed by each observer for himself.

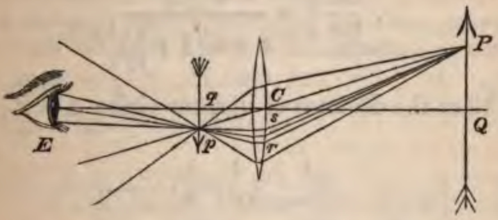
197. *Vision through a lens.*

Let PQ be a small luminous object, C the centre of a lens whose axis is CQ, E the centre of the pupil of an eye whose axis coincides with the axis of the lens. A pencil of



diverging from a point P of PQ falls upon the lens, and refraction may be considered as diverging from some t p in CP, or in CP produced, or as converging to some t p in PC produced. Thus pq an image of PQ is formed.

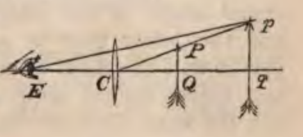




q be not less than the least distance of distinct vision of ye, then of the pencil diverging from p, the pupil selects tion prs which has been refracted *excentrically* through ens, and by this pencil the point p is visible. Thus the e pq will be seen by the eye.

OR. If PQ be very near to the principal focus of the so that the image pq may be very distant, the excentrical il pE by which the point p is seen may be considered to st of rays parallel to pC. (*Compare* Art. 115.)

98. *To find an expression for the visual angle under h a small object is seen through a lens.*



et EC be the axis of the lens, e eye, PQ the object, pq its e, $EC = x$, $CQ = u$, $PQ = y$, $= d$, $f =$ focal length of the $\angle pEq = \phi$, $\angle PEQ = \alpha =$ angle h PQ subtends at the eye.

[illegible]

[illegible]

$$= \frac{y}{u - x\left(1 - \frac{u}{f}\right)} \text{..........................(i).}$$

[illegible]

$$\tan x = \frac{y}{u - x};$$

$$\therefore \frac{\tan\alpha}{\tan x} = \frac{u - x}{u - x\left(1 - \frac{u}{f}\right)} \text{..........................(ii).}$$

If pq be at the distance of distinct vision $\Delta = Eq$ suppose, then

$$\frac{1}{\Delta - x} = \frac{1}{f} - \frac{1}{u} \text{..........................(iii).}$$

The results (i), (ii), (iii) enable us to discuss the apparent magnitude for different relative positions of the object.

If the rays emerge in a state of parallelism, then $u = -f$, the lens must be a convex one—i. e. f negative—and we shall have $\tan\phi = -\frac{y}{f}$ from (i). Hence the less f is, the greater will ϕ be,—and this suggests a reason why $\frac{1}{f}$ is called the *power* of a lens (Art. 104).

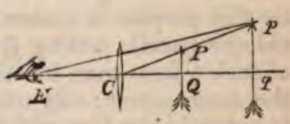
199. In the results of the previous article, if $\phi > \alpha$ the visual angle of the image is greater than that of the object, and the latter may be said to be magnified, and conversely. But it will be useful to compare the magnitudes of the image and object on the supposition that they are both observed at the same distance—that of distinct vision,—and we give the following

)**EFINITION.** When an object is seen through a lens the ıifying power of the lens is the ratio of the angle which mage subtends at the eye—to the angle which the object d subtend at the eye if it were in the position of the e and viewed directly.

'his mode of estimating magnifying power is equivalent mparing the linear magnitudes of the image and object.

'00. *To find the magnifying power of a lens* for given ions of the object and eye with respect to the lens.

'he object being supposed small the angle which pq ends at the eye $= \frac{pq}{Eq}$, and PQ ed at distance Eq would sub- an angle $= \frac{PQ}{Eq}$. If then m te the magnifying power of lens,



$$m = \frac{pq}{PQ} = \frac{Cq}{CQ} = \frac{v}{\omega}. \text{ But } \frac{1}{Cq} - \frac{1}{CQ} = \frac{1}{f}; = \frac{1}{v} - \frac{1}{\omega} = \frac{1}{f}$$

$$\therefore m = 1 - \frac{Cq}{f}. = 1 - \frac{v}{f}$$

'01. *To examine the values of* m *under different circum-es.*

n the figure of last Article

) If f be positive—or the lens concave—Cq is positive the image erect. Also Cq is $< f$, therefore m is < 1, or mage is diminished with respect to the object.

i) If f be negative—or the lens convex—Cq is positive gative, and the image erect or inverted, according as CQ ss or greater than the focal length of the lens without ence to sign.

[illegible]

Let $\left.\begin{matrix}f_0\\ y_0\end{matrix}\right\}\ \begin{matrix}f_e\\ y_e\end{matrix}$ be the focal lengths and half-breadths of the ject-glass and eye-glass respectively.

Join *Ab* cutting *pq* in *p*—through *p* draw a line parallel to c; then these two lines with the intercepted portions of the nses—(treated as straight lines),—will give us two similar iangles, whence we have

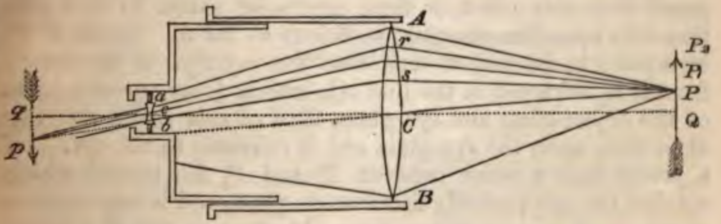
$$\frac{y_e - \rho}{f_e} = \frac{y_0 + \rho}{f_0},$$

$$\text{therefore} \quad \rho = \frac{f_0 y_e - f_e y_0}{f_0 + f_e},$$

$$\text{and} \quad \phi = \frac{\rho}{f_0} = \frac{f_0 y_e - f_e y_0}{f_0 (f_0 + f_e)}.$$

207. *Galileo's Telescope.*

ACB is a convex lens called the object-glass, fixed in a tube, and *acb* a concave lens called the eye-glass, fixed in another tube, which slides in the former,—the common axis of the tubes being the common axis of the lenses. The focal length of the object-glass is numerically greater than that of the eye-glass, and when the instrument is adjusted for viewing very distant objects, the distance between *C*, *c* the centres of the lenses, is the *difference* of the focal lengths.



If the axis of the Telescope be directed to a point *Q* of an object *PQ*, which is so distant that a pencil incident on the object-glass from any point of it may be considered to consist of parallel rays, the pencil from a point P after refraction through the object-glass converges very nearly to a point p in PC produced,—Cp being equal to the focal length of the object-glass—and thus pq an inverted image of PQ *would* be formed. Of the pencil converging to any point p of this image, the eye-glass (which is about the size of the pupil of the eye) selects the portion prs which has been refracted *excentrically* at the object-glass,—and since by the adjustment of the instrument pq is at the principal focus of the eye-glass, this portion of the pencil after *centrical* refraction through the eye-glass consists of rays parallel to cp, and suitable for giving distinct vision of the image of p formed by the eye-glass to an eye applied to the eye-glass.

Thus an *erect* image of PQ is seen through the Telescope —since the rays from the extremities of the object have not crossed each other before entering the eye.

208. *Field of View of a Galilean Telescope.*

If P_1 be a point in the object such that the ray P_1A of a pencil from it is refracted in the line Aa, which joins *corresponding* parts of the object-glass and eye-glass, then of the pencil from this point, or from any point nearer to Q, a portion falls upon the eye-glass sufficient to fill it. Again if P_2 be a point in the object such that the ray P_2A of the pencil from it is refracted in the line Ab, which joins *opposite* parts of the object-glass and eye-glass, then of this pencil this ray alone falls upon the eye-glass and is refracted to the eye. Of a pencil from a point between P_1 and P_2 the portion which reaches the eye partially fills the eye-glass, and is less and less as the point is more distant from Q; also, no ray of a pencil from a point more distant from Q than P_2 is refracted by the eye-glass. Hence there will be a *ragged edge* to the field of view, which in this Telescope is incurable—because the image formed by the object-glass is *virtual*, and therefore cannot be limited by a stop.

Obs. The vision through the telescope will be most distinct when the refraction through the eye-glass is *centrical*, and hence the size of the eye-glass ought to be nearly the same as that of the pupil of the eye:—if it be much larger and the eye be not applied at its centre, the refraction through it will be excentrical, and the distortion and chromatic dispersion increased.

209. *Def.* When an object is viewed by a telescope the *magnifying power* of the telescope is estimated by the ratio of the angle which the image seen subtends at the eye to the angle which the object would subtend at the eye if viewed directly.

210. *To find the magnifying power of the Astronomical Telescope—or of Galileo's Telescope.*

Since the point p of the image pq (figs. Arts. 204, 207) is seen by a pencil whose axis after refraction through the eye-glass is parallel to pc, and the point q by a pencil whose axis is qc, the image of PQ subtends at the eye an angle pcq. And PQ would subtend at the eye an angle PcQ,—or PCQ since the object is very distant.

Hence

$$\text{magnifying power} = \frac{\angle pcq}{\angle PCQ} = \frac{pcq}{pCq}$$

$$= \frac{\tan pcq}{\tan pCq} \text{ approximately,}$$

$$= \frac{Cq}{cq}$$

$$= \frac{\text{focal length of the object-glass}}{\text{focal length of the eye-glass}}.$$

Obs. This result gives *approximately* the magnifying power at any point of the field of view:—though it is strictly true near the centre of the field of view only.

Further, the instruments are supposed to be in adjustment, so that any pencil from a very distant point emerges from the eye-glass as a pencil of parallel rays: *practically* the eye-glass must be pushed in a little, in the case of either telescope, to suit a short-sighted eye,—so that from this cause the magnifying power will be slightly different for different observers.

[illegible] from any point of it may be considered to [illegible] rays, the pencil from a point P after refr[illegible] object-glass converges very nearly to a poi[illegible] —Cp being equal to the focal length of th[illegible] and thus pq an inverted image of PQ wo[illegible] On the pencil converging to any point p [illegible] the eye-glass (which is about the size of the p[illegible] [illegible] the portion prs which has been r[illegible] at the object-glass,—and since by the adj[illegible] [illegible] pq is at the principal focus of the ey[illegible] the portion of the pencil after concave refraction [illegible] the eye-glass consists of rays parallel to cp, and suit[illegible] giving distinct vision of the image of p formed by [illegible] glass to an eye applied to the eye-glass.

Thus an erect image of PQ is seen through the T[illegible] —since the rays from the extremities of the object [illegible] crossed each other before entering the eye.

[illegible] Field of View of a Galilean Telescope.

If [illegible] the ray [illegible]

[illegible]

Obs. The [illegible] hence the size of the eye-glass [illegible]

212. *Field of View* of Newton's Telescope.

If P_1 be a point in the object such that the ray of a pencil from it which, after reflexion at ACB, is incident on the small mirror at D, is reflected by DF in Db the line joining opposite parts of the small mirror and eye-glass, then of a pencil from P_1—or from any point of the object nearer to Q than P_1—every ray which falls on the small mirror is refracted by the eye-glass to the eye. Again if P_2 be a point in the object such that the ray of a pencil from it which is incident on the small mirror at F is reflected in Fb the line joining corresponding parts of the mirror and eye-glass, then of the pencil from P_2 this ray alone reaches the eye. Points between P_1 and P_2 appear more and more faint the farther they are from Q,—and points at a greater distance from Q than P_2 is are invisible.

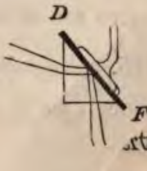
213. *Magnifying power* of Newton's Telescope.

Let $p'q'$ (fig. Art. 211) be the virtual image of PQ formed by the large mirror: pq is similar and equal to $p'q'$. Now pq subtends through the eye-glass the angle $\frac{pq}{cq}$, and PQ subtends to the naked eye the angle POQ or $\frac{p'q'}{Oq'}$.

Hence

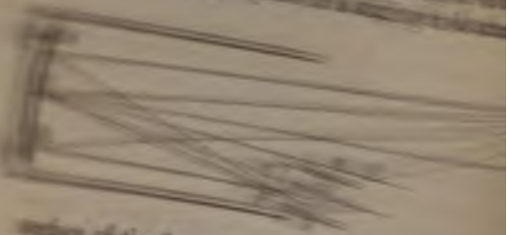
$$\begin{aligned}\text{magnifying power} &= \frac{pq}{cq} \cdot \frac{Oq'}{p'q'} \\ &= \frac{Oq'}{cq} \\ &= \frac{\text{focal length of large reflector}}{\text{focal length of eye-glass}}.\end{aligned}$$

Note. The form of the small mirror must be an ellipse in order that it may intercept as little of the incident light as possible, and just reflect all the light incident upon the curved mirror. A rectangular prism of glass is sometimes used instead of a plane mirror; the passage of a pencil through it is indicated in the figure.



[illegible] Herschel's [illegible]

[illegible]



surface of the [illegible] that of the large [illegible] equally inclined to CO.

If the axis of the [illegible] an object PQ, which is so distant [illegible] ABC from any point of it [illegible] parallel [illegible]

[illegible]

[illegible] image of PQ is seen through the [illegible]

[illegible] The construction of the reflecting telescope [illegible] proposed by Le Maire in the early part of [illegible] Sir W. Herschel was the first who made [illegible]

Magnifying power of Herschel's Telescope.

$\angle pCO = \angle PCO$, and $\angle qCO = \angle QCO$ (Art. [illegible]);

$\therefore \angle pCq = \angle PCQ$.

Now pq viewed by the eye-glass subtends the angle pcq, and PQ viewed by the naked eye subtends the angle PCQ, or pCq;

$$\therefore \text{ magnifying power} = \frac{\angle pcq}{\angle pCq} = \frac{Cq}{cq}$$

$$= \frac{\text{focal length of mirror}}{\text{focal length of eye-glass}}.$$

216. The principle of Herschel's Telescope is the same as that of Newton's—the only object of the plane reflector in the latter being to throw the image unaltered in form into another position where it may be more conveniently viewed. The speculum in Herschel's Telescope is designedly of large aperture, so that when a faint star is observed a pencil sufficiently large to make it visible may be received by the speculum and reflected to the eye.

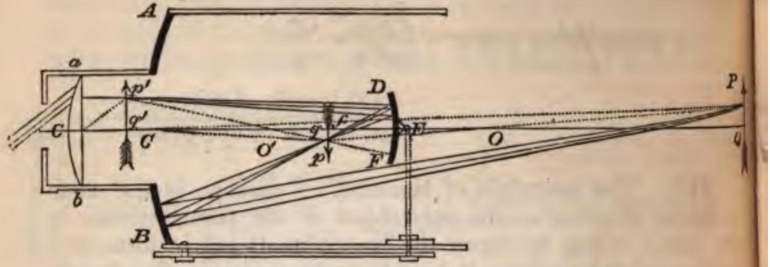
The advantage of Herschel's construction over Newton's arises from there being no part of the incident pencils stopped by the back of the small mirror, and no loss of light from a second reflexion.

The ragged edge of the field of view in Herschel's Telescope may be remedied by a stop placed at the principal focus of the eye-glass—and the angular diameter of the field of view will be the angle which the aperture of the stop subtends at the centre of the face of the speculum—or *approximately*, the angle which the breadth of the eye-glass subtends at the same point.

A similar method of suppressing the ragged edge and estimating the field of view, will apply to Newton's Telescope.

217. *Gregory's Telescope.*

ACB, *DEF* are two concave spherical reflectors, or specula, with a common axis *CE*, which is the axis of a tube at the extremity of which *ACB* is placed,—this mirror being much larger than *DEF* and of larger radius. The concavities of the mirrors are turned towards one another, and the principal focus of *ACB* is between the centre *O'* and principal focus *f* of *DEF*. In a tube which is fixed in an aperture at the centre of ACB is a convex eye-glass acb, the axis of the eye-glass coinciding with that of the reflectors.

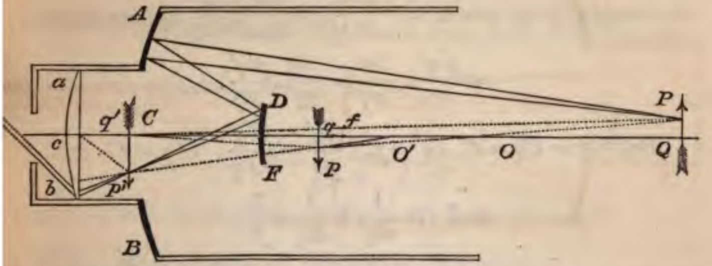


If the axis of the reflectors be directed to a point Q of a very distant object PQ, a pencil from any point of it P after reflexion at ACB converges very nearly to a point p in the straight line produced which joins P with O the centre of ACB, and thus pq a *real* inverted image of PQ is formed at the principal focus of ACB. Since this image is between the centre and principal focus of DEF, a pencil diverging from any point p after excentrical reflexion at DEF converges to a point p' in the straight line produced which joins p with O', the centre of DEF (Art. 115), and thus $p'q'$ an erect image of PQ is formed. This image being at the principal focus of the eye-glass is in a suitable position for being distinctly seen through the eye-glass,—and thus an *erect* image is seen through the telescope.

Obs. In Gregory's and Cassegrain's telescopes the adjustment for different eyes is performed by shifting the small mirror by means of a screw, as is shewn in the figure,—in each of the other telescopes, it is performed by moving the eye-glass backward or forward.

218. *Cassegrain's Telescope.*

ACB is a concave and DEF a convex spherical reflector with a common axis CE, which is the axis of a tube at one extremity of which ACB is placed. The mirror ACB, which is much larger and of greater radius than DEF, has its concavity turned towards the convexity of DEF, and the principal focus of ABC is between E and f the principal focus of *DEF*. In a tube which is fixed in an aperture at the centre of *ACB* is a convex eye-glass *acb*, the axis of the eye-glass being that of the reflectors.



If the axis of the reflectors be directed to a point Q of a very distant object PQ, a pencil from any point P, after reflexion at ACB, converges very nearly to a point p in the line which joins P with O the centre of ACB, and then pq an inverted image of PQ would be formed at the principal focus of ACB. Since this image is between DEF and its principal focus, the pencil converging to any point p after excentrical reflexion at DEF converges to a point p' in the straight line produced which joins O' the centre of DEF with p, and $p'q'$ an inverted image of PQ is formed. This image being at the principal focus of the eye-glass is in a suitable position for being distinctly seen through the eye-glass, and thus an *inverted* image is seen through the telescope.

219. *Magnifying power* of Gregory's Telescope.

Take the figure of Art. 217.

Let $\left.\begin{array}{l} Cq = Oq = F \\ Ef = O'f = f \end{array}\right\} =$ focal lengths of $\left\{\begin{array}{l}\text{large}\\ \text{small}\end{array}\right.$ mirror.

$cq' = f_e$ = focal length of the eye-glass, numerically.

$qf = x$ = distance between the principal foci of the two mirrors.

Let Q be the point of the object to which the axis of the Telescope is directed,—the image of any other point P by reflexion at ACB is at p—and the image of p formed by reflexion at DEF is at p'—the straight line pp' passing through O' the centre of DEF.

The portion PQ of the object would subtend at the eye the angle POQ, and the image of PQ subtends at the eye the angle $p'cq'$. Hence

$$m = \text{magnifying power} = \frac{\angle p'cq'}{\angle POQ} = \frac{\tan p'cq'}{\tan POQ} \text{ approximately,}$$

$$= \frac{p'q'}{cq'} \cdot \frac{Oq}{pq} = \frac{Oq}{cq'} \cdot \frac{p'q'}{pq}.$$

But $Oq = Cq = F$, $cq' = f_e$, $\dfrac{p'q'}{pq} = \dfrac{O'q'}{O'q}$, $O'q = f - x$,

$$\text{and } \because \frac{1}{Eq'} + \frac{1}{Eq} = \frac{1}{f},$$

$$\text{or } \frac{1}{O'q' + 2f} + \frac{1}{f + x} = \frac{1}{f}, \text{ whence } O'q' = \frac{f(f - x)}{x};$$

$$\therefore \frac{p'q'}{pq} = \frac{f}{x},$$

$$\text{and } m = \frac{F.f}{f_e \cdot x}.$$

Obs. This expression is strictly accurate at the *centre of the field only*, but may be taken as approximately true over the whole.

Note. The expressions for m in this article and the following one are equally true for Cassegrain's Telescope.

As the adjustment of the Telescope for different eyes is effected by moving the small mirror by means of a tangent screw, the above expression is a very convenient one, since it involves x, the variable quantity in the instrument, in a very simple form.

220. An approximate expression for the magnifying power may be obtained independent of x.

Since Cq' is in general small compared with Oq, we will regard q' as coincident with C, approximately.

We then have $\dfrac{1}{EC} + \dfrac{1}{Eq} = \dfrac{1}{f}$,

$$\text{or } \frac{1}{F + f + x} + \frac{1}{f + x} = \frac{1}{f},$$

bolic or slightly hyperbolic. The defects of the single eye-lens which are lessened by a compound eye-piece are the same as have been mentioned in the *Astronomical Telescope.*

226. In the Astronomical Telescope since the pencils pass centrically through the object-glass and excentrically through the eye-glass, the *field of view* depends only on the aperture of the eye-glass;—the aperture of the object-glass affecting only the *brightness* of the field.

In Galileo's Telescope, on the contrary, where the refraction through the object-glass is excentrical, the *field of view* depends upon the aperture of the object-glass—and this is the reason why this telescope is not so generally used for astronomical purposes as for a perspective or opera-glass where small magnifying power is required. For with a high magnifying power, and a field of any considerable extent, the extreme pencils would be refracted by the object-glass at such a distance from its axis as to make their chromatic dispersion unpleasant and with difficulty diminished. Moreover, the brightness of the field in Galileo's telescope is nearly the same as that of the object, for (fig. Art. 207),

the breadth of the visual pencil at the object-glass *rs*
: its breadth at the eye-glass
:: focal length of object-glass
: focal length of eye-glass,

i.e. in the ratio of the magnifying power to unity; in other words, the quantity of light received by the eye from any small area of the object is greater than what would be received from the same area without the intervention of the instrument, in the same ratio that the area is magnified—and therefore its apparent brightness is unaffected.

This telescope exhibits objects erect,—which is of great advantage for the purposes for which it is generally employed.

227. *Specula for Reflecting Telescopes.*

The alloy used as a speculum metal is composed of 126·4 parts of copper to 58·9 of tin—from which proportions however different experimenters deviate more or less.

$$Cq = F,\ Ef = f,\ fq = x = \frac{f^2}{F}, \text{ approximately (Art. 220).}$$

$$\text{Now } \frac{\phi}{\angle aO'c} = \frac{\sin\phi}{\sin CO'p} = \frac{O'p}{Cp}$$

$$= \frac{O'q}{Cq} = \frac{f-x}{F}, \text{ nearly.}$$

$$\text{Also } \angle aO'c = \frac{y_e}{O'c} = \frac{y_e}{f_e + O'q'},$$

$$\text{but } O'q' = \frac{f(f-x)}{x} = \frac{F}{f}(f-x) \qquad \text{(Art. 219);}$$

$$\therefore \phi = \frac{f-x}{F} \cdot \frac{y_e}{f_e + \frac{F}{f}(f-x)} = \frac{f \cdot y_e}{F(F+f_e)}, \text{ nearly,}$$

since x is small compared with F and f.

(ii) If the field be limited by the small mirror—let the ray PC after reflexion at C fall upon the edge of the small mirror, then if the pencil which falls upon the object-mirror from any point nearer to Q than P is, more than one-half reaches the eye.

If $\phi_1 = PCQ$ be the angular radius of the field of view thus defined, we have

$$\phi_1 = \frac{y_m}{F+f+x} = \frac{y_m}{F+f}, \text{ nearly.}$$

Cor. If the breadths of the eye-glass and small mirror are such that the field of view limited by them severally is the same, then $\phi = \phi_1$ and

$$\frac{y_m}{F+f} = \frac{f \cdot y_e}{F(F+f_e)};$$

$$\therefore \frac{y_m}{y_e} = \frac{f(F+f)}{F(F+f_e)} = \frac{f}{F}, \text{ nearly,}$$

since f_e and f are small compared with F.

an adjustment which it costs some trouble to secure, and which is very easily deranged; it is also readily affected by a very slight tremor of the stand of the telescope:—but notwithstanding these disadvantages it is in great favour with many observers from its compactness and the convenience resulting from its shewing objects erect.

The Cassegrain construction is not much used.

229. A large telescope has generally a small field of view, and there is often difficulty in directing it to a proposed object: to diminish this, a small telescope, called a *finder*, of small power and large field of view is often attached externally to the tube of the large telescope; the axes of the two being coincident. A considerable extent of space being thus within the field of the finder the instrument can be moved till the object proposed coincides with the centre of the field of the finder—i. e. at the intersection of its cross-wires—and it is then at the centre of the field of the large one.

230. *Object Glasses.*

The compound object-glass commonly used in a refracting telescope consists of a lens of crown glass in contact with a lens of flint glass. The conditions which the combination has to fulfil are that it shall be *achromatic* for given kinds of light—(see Art. 178)—and also *aplanatic* or free from spherical aberration. (See Art. 132).

Achromatic object-glasses have also been constructed of three lenses in contact, consisting of a concave lens of flint glass between two convex lenses of crown glass. The conditions of achromatism can thus be satisfied for more species of light than in the former case: the forms of the lenses will be determined on principles similar to those just referred to. Such object-glasses are now not frequently employed in consequence of the difficulty of centering the lenses so that their axes may exactly coincide. Object-glasses of three lenses have occasionally been made in which the middle one consisted of some liquid, but they have been found not to be very durable in consequence of a chemical change in the liquid.

the references given in this and
e student may consult *Lehrbuch der*
I. C. E. Schmidt, for a method propo
g into account the *thickness* of the lens
object-glass,—and Grunert, *Optische Untersu*
Camb. Phil. Trans. Vol. VI.

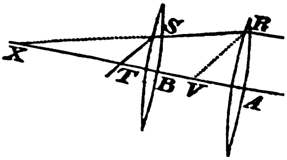
231. *Eye-pieces.*

The defects of a single eye-lens—referred to i
viz. indistinctness, linear distortion, as well as that of
and chromatic dispersion—are diminished by em
compound eye-piece. Those in most general use are
ghens' Eye-piece, (ii) *Ramsden's* Eye-piece: they
known as the *negative* and the *positive* eye-pieces resp

From the expression for tan η (Art. 137) we see t
distortion produced by a single lens $\propto \frac{1}{f^2}$. With a
diminishing the distortion, Huyghens proposed to co
an eye-piece of two separated convex lenses—dividing e
between the two the deviation produced in an excen
pencil.

232. *To find the distance between two lenses in order the excentrical pencil incident parallel to the axis may suffer equal amount of deviation at each lens.*

Let $QRST$ represent the course of the *axis* of the pencil, f_1, f_2 the *numerical* focal lengths of the lenses A, B, which we will suppose *convex* ones, $AB = a$.



Then using first approximations only, we have

$$\text{deviation at } R = \angle RXA,$$

$$\ldots\ldots\ldots\ldots \quad S = \angle XST = \angle STB - \angle RXA,$$

if these be equal, we get $\angle STB = 2\angle SXB$,

and the angles being small, $\tan STB = 2 \tan SXB$;

$$\therefore\ BX = 2 \,.\, BT.$$

Also, $\frac{1}{BT} - \frac{1}{BX} = \frac{1}{f_2}$; $\therefore BX = f_2$.

But $BX = AX - AB = f_1 - a$;

$$\therefore a = f_1 - f_2 \text{ the distance required.}$$

The construction adopted by Huyghens in consistence with this condition was an eye-piece of two convex lenses whose focal lengths are in the ratio of 3 : 1, the less powerful lens being placed as a *field*-glass—i. e. nearest the object-glass of the telescope—and the distance between the lenses being the difference of their focal lengths.

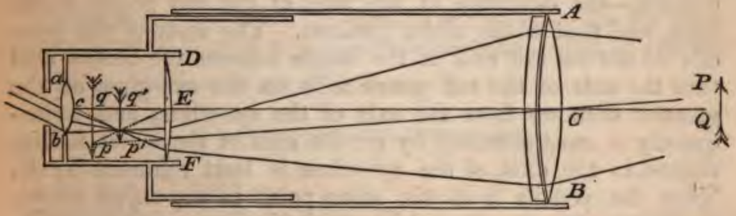
It is a remarkable coincidence—undesigned by the inventor—that if the lenses be of the same material, this construction fulfils simultaneously the condition of achromatism of an excentrical pencil, viz.

$$a = \tfrac{1}{2}(f_1 + f_2) \text{ (Art. 181).}$$

The Huyghenian—or *negative*—eye-piece is therefore *achromatic.*

233. *Vision through an Astronomical Telescope with a Huyghens' eye-piece.*

Let ACB be the object-glass of an Astronomical Telescope directed to a very distant object PQ: DEF the first



lens, or *field-glass*, and *acb* the second lens, or *eye-glass*, of a Huyghens' eye-piece, the focal length of DEF being three times that of *acb*, and the distance *Ec* being the difference, or semi-sum, of the focal lengths without reference to sign. A pencil from a point P of the object after refraction through the object-glass would converge very nearly to a point p, Cp being equal to the focal length of the object-glass,—but being excentrically refracted by the field-glass converges to a point p' in Ep, and thus $p'q'$ an inverted image of PQ is formed. The position of the eye-piece is such that q' is the bisection of Ec, and therefore this image is at the principal focus of the eye-glass. Hence a pencil from any point p' of the image after excentrical refraction at the eye-glass consists of rays parallel to $p'c$, and suitable for giving distinct vision of the image formed by the eye-glass to an eye applied to the eye-glass. Thus an inverted image of PQ is seen through the telescope.

234. The compensation between the two lenses which renders Huyghens' eye-piece achromatic admits of a simple general explanation.

The deviation of the axis of a pencil of light produced by a convex lens is greater the greater the distance from the axis of the lens at which the axis of the pencil is refracted: for this axis is refracted in the same degree as it would be by a prism whose surfaces touch the lens at the points where the axis of the pencil is incident and emergent, and therefore the deviation is greater as the refracting angle of such prism is greater (Art. 92). Now when a pencil of light refracted by the object-glass falls on the field-glass, it is separated by it into a series of coloured pencils whose axes follow different courses,—the deviation of the axis of the red pencil being least, and that of the violet greatest. The axes of the pencils do not cut the axis of the lenses *between* the lenses, and thus the axis of the red pencil falls on the eye-glass at the greatest distance from the axis of the eye-glass, and consequently is most refracted by it; the axis of the violet falling nearest to the axis of the eye-glass is least refracted by it. Thus the pencils from the same point in the object which are least and most refracted by the field-glass are respectively most and least refracted by the eye-glass, and consequently, by a proper selection of the lenses, may be parallel when *they enter* the eye.

235. The position of the eye-piece is determined by the dition that a direct pencil after refraction through the d-glass may have the principal focus of the eye-glass for geometrical focus—i. e. a direct pencil must at incidence on *EF* be converging to a point q such that after refraction ough *DEF* it converge to q', the principal focus of *acb*, nce q' must be the middle point of *Ec*, and *Eq* must there-e (Art. 99) be equal to half the focal length of *DEF*, or ee-fourths the distance *Ec*.

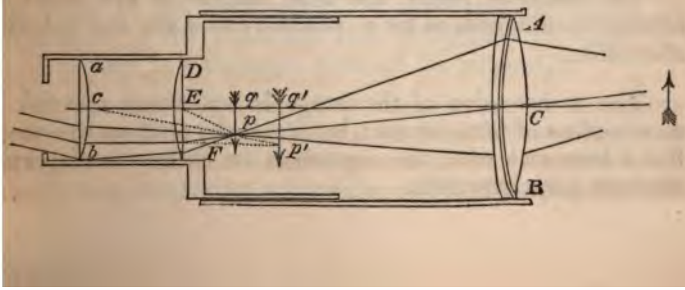
The focal lengths and position of the lenses of a Huy-ens' eye-piece being determined, the *forms* of the lenses to be chosen with reference to the use of the telescope. e conditions for diminishing *distortion*, *indistinctness*, and *rvature* of the field being different, it is a matter of judg-nt which of these defects are most to be avoided. Accord-g to Mr Coddington (*Optics*, q. v.) these defects will be viated as far as possible by making the field-glass a *menis-*, having its radii in the ratio of 11 : 4; and the eye-glass *rossed lens*, the radii being as 1 : 6,—the forms indicated in figure.

236. *Ramsden's Eye-piece.*

This eye-piece, sometimes called the *positive* eye-piece, is ombination of two separated convex lenses, for the purpose diminishing the effects of spherical aberration—which can effected better by two lenses than by one of equivalent wer, because more disposable quantities are thus introduced o the calculations of the forms of the lenses.

The lenses are taken of *equal* focal length, and the dis-ce between them is *two-thirds* of the focal length of either.

This eye-piece is not achromatic.



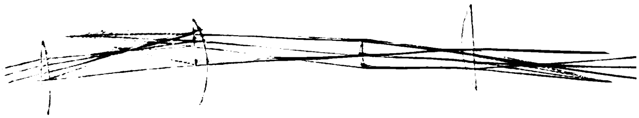
passage of a pencil through an Astronomical provided with a Ramsden's eye-piece is indicated in

of the eye-piece, when in adjustment, is condition that a direct pencil at incidence must be diverging from a point q such on through DEF it may diverge from a point with the principal focus of acb: hence since of acb, therefore $Eq' = \frac{1}{4}$ of the same fore (Art. 99) $Eq = \frac{1}{2}$ of the same.

considerations which lead to the *forms* of the are si to those mentioned in the case of Huyghens' pieces lenses are generally of the forms in the fig the s being plano-convex, the eye-glass plane.

237. *The Erecting Eye-piece.*

The inversion of the image by an Astronomical Tele when furnished with either of the eye-pieces already desc renders it unsuitable for viewing terrestrial objects remedy this, an *Erecting Eye-piece* of four lenses is com used. The manner in which an object is seen through scope with this eye-piece will be sufficiently understood the course of a pencil traced in the figure.



The distances, forms, and focal lengths of the adjusted to diminish as far as possible chromatic and aberration.

The re-inversion of the image is sometimes an eye-piece of three lenses; but in all the erecting that I have met with, the correction for chromatic has been very

238. In the description of the eye-pieces they have been
pposed to be employed in an Astronomical Telescope. The
me eye-pieces are however employed with the Reflecting
elescopes. In Galileo's telescope a single eye-lens is gene-
lly used because the refraction through it is centrical. The
vestigations of the field of view will still be true in tele-
opes with compound eye-pieces, if the field-glass of the eye-
ece be used in them instead of the eye-lens. The determi-
tion of the magnifying powers will also hold good, if the
mple eye-lens be supposed such as will refract an excentrical
ncil in parallel rays at the same inclination to the axis of
e lenses as it has at emergence from the eye-glass (Art. 138)
i. e. if we substitute the *equivalent lens* for the eye-piece.

On the theory of eye-pieces, see Grunert, *Optische Unter-*
chungen, Theil III., and two memoirs by Mr Airy in the
mbridge Phil. Trans. Vol. II. and III.

239. Since the field of view of a telescope is of finite
tent, it is necessary to have certain points in the field to
ich an object observed for the purpose of measurement may
referred. This is in general attained by fixing in the
be of the telescope *a set of fine parallel threads* in a plane
rpendicular to the axis of the lenses, and *one or more*
reads at right angles to this set, which if placed at one of
e images formed by the telescope are, like that image, dis-
ctly visible through the eye-glass.

In Huyghens' Eye-piece the wires would be placed at the
incipal focus of the eye-glass, and therefore would be dis-
rted by excentrical refraction through that lens alone; while
e image seen would be distorted by excentrical refraction
rough the field-glass and eye-glass, and consequently in a
fferent degree from the wires. In this case the position of
y point of the field would be estimated incorrectly by
ferring it to the wires, and thus Huyghens' Eye-piece can
ver be used in a telescope intended for measuring.

In Ramsden's Eye-piece the image given by the object-
lass is formed in front of the field-glass, and at this image
e wires are placed. The image and the wires are thus each
en by two excentrical refractions, and are therefore distorted

243. The *Dynamometer* above referred to (Art. 241), consists mainly of a slip of mother-of-pearl with a scale of tenths of an inch engraved on it, on which the bright spot may be received so that its diameter may be read off at once. When greater accuracy is required, an eye-piece with a *divided field-glass* is employed, one half of this lens remaining fixed and giving a bright circular image of the object-glass in a permanent position—the other half lens can be moved by a screw transversely to the axis of the instrument, and the bright image formed by it is brought into contact with the other fixed image —first on one side of it and then on the other: the difference of the readings of the graduated head of the screw in these two positions affords a means of determining the diameter of the bright image.

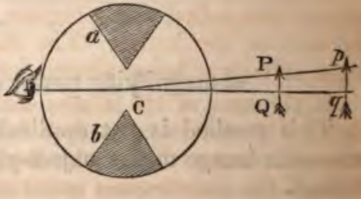
244. *Microscopes.*

Some objects are so minute that when they are viewed by the naked eye at the least distance of distinct vision, the distances of their parts subtend no appreciable angles, and therefore cannot be discerned. In these cases it is advantageous to view an *image* of the object, instead of the object itself,—and an instrument for this purpose is called a *microscope.*

Microscopes are called *simple* or *compound,* according as a real image of an object viewed by them *is not* or *is* formed.

245. A *single lens,* or a *sphere,* forms a simple microscope.

If an object be placed nearer to a convex lens than its principal focus, an erect and magnified image of it may be seen by an eye on the axis of the lens (Art. 201).

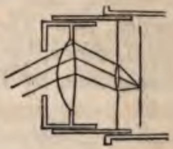


Also if a small object PQ be placed nearer to the centre of a refracting sphere than its principal focus, a pencil diverging from a point P will after direct refraction through the sphere *diverge* very nearly from some point p in CP produced, and pq an erect image of PQ is thus formed. This image, if its distance from an eye close to the sphere be not less than the least distance of distinct vision, may be seen by the eye by direct pencils.

The distinctness will be much improved, if the visual pencils be restricted to pass nearly through the centre of the sphere—which can be effected by filling up a groove a, b, leaving only a small circular opening at the centre of the sphere—the effect of spherical aberration will be thus almost entirely obviated.

246. Since the minimum aberration for parallel rays is less for substances of high refractive power, it is of great advantage to construct lenses for this purpose of substances for which μ is large;—for example, there is much less aberration in a lens of zircon $(\mu = 2)$ than in a lens of crown-glass. The diamond has a still higher refracting power than this, which combined with its low dispersive power makes it the most desirable substance to be used for this purpose.

A simple microscope preferable to a single lens is composed of two convex lenses separated by a small distance on a common axis. If an object be placed nearer to the first lens than its principal focus, so that a virtual image of it may be formed by each lens, the image formed by the second lens will be distinctly seen by an eye whose axis is the axis of the lenses, and whose distance from this mage is not less than the least distance of distinct vision. This is the principle of Wollaston's *Microscopic Doublet.*

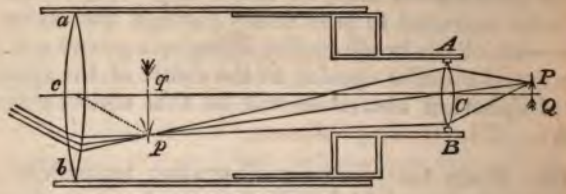


247. *The compound refracting microscope is an Astronomical Telescope adapted for viewing near objects.*

ACB is a convex lens called the object-glass, and acb a convex lens called the eye-glass, fixed in a tube whose axis is he axis of the lenses. The distance of the centres of the enses admits of being altered for the purpose of adjustment.

If the axis of the lenses be directed to a point Q in an object PQ which is farther from the object-glass than its principal focus, the pencil from a point P after refraction hrough the object-glass converges very nearly to a point p

in PC produced, and thus pq an inverted image of P
formed. The position of the eye-glass is such that t
image is at its principal focus, and therefore the pencil fr
any point p of the image consists after excentrical refract



through the eye-glass of rays parallel to pc, and suited
give distinct vision to an eye applied to the eye-glass, a
thus an inverted image of the object is seen through t
microscope.

248. Compound object-glasses and eye-pieces are co
monly employed for reasons similar to those which ren
them necessary in telescopes.

For a full account of the microscopes of different make
the mode of mounting and illuminating the object under o
servation, &c., and the recent development of microsco
science, see *The Microscope, its History, Construction, a*
Applications, by Jabez Hogg; also *The Microscope and*
Revelations, by Dr Carpenter.

249. *Def.* The magnifying power of a compound
croscope may be estimated by the ratio of the angle which
image seen subtends at the eye to the angle which the obj
would subtend at the eye, if placed at the distance of disti
vision and viewed directly.

Thus if $K =$ distance of distinct vision,
$CQ = u$, f, F focal lengths of ACB, acb without reference
sign, we have

$$\text{angle subtended by the image} = \frac{pq}{cq} = \frac{pq}{F},$$

$$\text{angle which object would subtend at distance } K = \frac{PQ}{K};$$

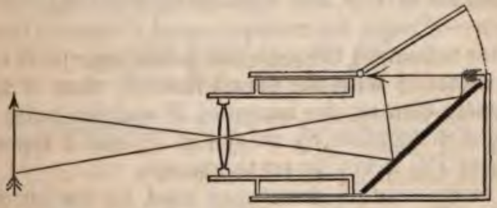
$$\therefore\ m = \frac{K}{F} \cdot \frac{pq}{PQ} = \frac{K}{F} \cdot \frac{Cq}{u}.$$

But numerically $\frac{1}{Cq} + \frac{1}{u} = \frac{1}{f}$; $\therefore\ Cq = \frac{fu}{u-f}$;

$$\therefore\ m = \frac{Kf}{F(u-f)}.$$

250. *The Camera Obscura.*

If in an aperture in the wall of a darkened room there be inserted a single convex lens, or a combination of lenses of considerable negative focal length, a real image of external objects is formed at a distance from the lens. If this image be received on a screen, either directly or after the direction of the pencils has been altered by reflexion at a plane mirror, an inverted picture of external objects is visible.



The annexed diagram represents a box constructed on the same principle, the image formed by the lens being received after reflexion at a plane mirror, on a piece of oiled paper, or ground glass, from which extraneous light is shaded by a lid which can move up and down.

251. If an object be placed before a convex lens, or combination of lenses, at a distance a little greater than that of the principal focus, and be illuminated by the sun or a powerful artificial light, a real inverted and magnified image of the object is formed, and if received on a screen in a darkened room will be seen as a picture on the screen.

This is the principle of the *Solar Microscope* and the *Magic Lantern*.

14—2

[illegible]

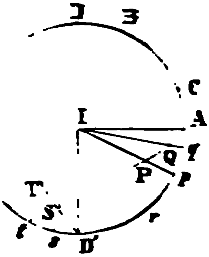
[illegible]

[illegible] will shew that the [illegible] the same as the projection [illegible]

[illegible] the edges of the prism [illegible] according to which light [illegible] them, meet the surface of a sphere [illegible] *B*, [illegible] *D*. Then [illegible] is [illegible] of the great circle [illegible]*D*, and

$$AC = \frac{\pi}{2} = BD,\quad BC = \frac{\pi}{4}.$$

Also let radii parallel to the axis of a pencil before and after refraction into



the prism, before and after refraction out of the prism, meet the sphere in P, Q, S, T, these radii being drawn in the direction opposite to that in which the light proceeds.

Let great circles through I and these points meet the circle $ABCD$ in p, q, s, t respectively. Produce DI to D' and join IA, ID'. Draw the great circles AQP, $D'ST$, and also the great circle Ir through the direction of the axis of the pencil after one reflexion. Then

$$Bq + Br = \pi, \text{ or } Aq + Ar = \frac{\pi}{4},$$

$$Cr + Cs = \pi, \text{ or } D's + Ar = \frac{\pi}{4};$$

$$\therefore Aq = D's.$$

$$\text{Also } IQ = IS, \text{ (Art. 78)}; \ \therefore \angle QAI = \angle SD'I,$$

$$\text{and } \frac{\sin TD'}{\sin SD'} = \mu = \frac{\sin PA}{\sin QA};$$

therefore the triangles IPA, ITD' are equal in all respects;

$$\therefore IT = IP \text{..........................(i)},$$

$$\text{and } \angle TID' = \angle PIA;$$

$$\therefore \angle TIP = \frac{\pi}{2} \text{........................(ii)}.$$

From (i) it appears that the axis of a pencil refracted and reflected by the prism of a Camera Lucida has at incidence and emergence the same inclination to any edge of the prism, and from (ii) it appears that planes parallel to the edges of the prism drawn through the axis of the pencil at incidence and emergence are perpendicular to each other. The effect of the prism therefore is merely to turn through 90°—about an axis parallel to the edge of the prism—the plane of the axis of any pencil, whilst in this plane the axis preserves the same direction as before relatively to the same edge.

Hence the picture seen on the screen pq is the same as the projection of the object PQ (fig. Art. 252) upon a plane parallel to AB.

Also α and δ being small, we have approximately

$$\delta = -\alpha^2 \tan\frac{\theta}{2},$$

the required correction to the reading of the limb.

If n be the number of seconds in the angle α, the correction in seconds $= -n^2 . \sin 1'' . \tan\frac{\theta}{2}$.

257. *The Reflecting Goniometer.*

The goniometer is an instrument for measuring the angle between two plane faces of a crystal, and consists of a circle of metal turning about an axis perpendicular to its plane. The rim of the circle is graduated, and is read by a pair of verniers in opposite positions. Let the crystal be attached to the circle, so that the plane of the latter is perpendicular to the intersection of the faces of the former whose inclination is required. Bring the circle into such a position that the image of a well-defined straight line perpendicular to the plane of the circle formed by reflexion at one of the faces of the crystal coincides with another well-defined straight line which is seen directly, and read the verniers:—window-bars will answer the purpose of these straight lines very well. Turn the circle until a similar coincidence is made between the same straight line seen directly, and the image of the other formed by reflexion at the other face of the crystal, and read the verniers again. The semi-sum of the differences of the two readings of each vernier is the angle through which the circle has been turned, and is equal to the angle between the normals to the two faces of the crystal and supplemental to the inclination of the two faces.

For a full description of the construction and use of this valuable instrument, which was invented by Dr Wollaston, see the article *Crystallography* in the *Encyclop. Metrop.*

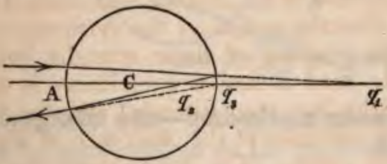
CHAPTER XI.

OF THE RAINBOW, &c.

258. If a pencil of light be refracted into a sphere, when t is incident on the interior surface of the sphere, a portion f it emerges and another portion is internally reflected; this atter portion being again incident on the interior surface is artially reflected and partially refracted; and so on coninually.

The intensity of light in the pencils which thus succesively emerge rapidly decreases.

259. Let C be the centre of a sphere of water, the re-



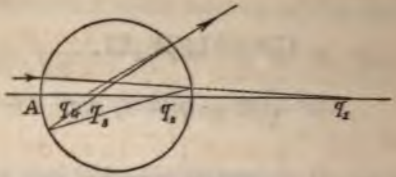
ractive index of which out of air for rays of mean refrangiility is 1·335, or $\frac{4}{3}$ nearly. Let a pencil of parallel rays of omogeneous light whose axis is in direction AC be incident irectly on the sphere.

Take $Cq_1 = 3AC$, $Cq_2 = \frac{3}{5}AC$, $Cq_3 = AC$, (Arts. 29, 24),

hen $q_1 q_2 q_3$ are the geometrical foci of the pencil when reracted into the sphere, reflected at the internal surface and mergent from the sphere respectively.

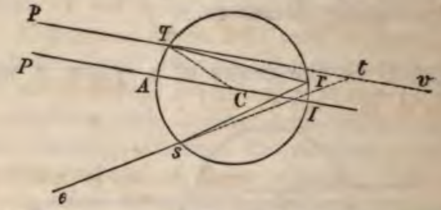
Again, if we take

$$Cq_1 = 3AC,\ \ Cq_2 = \frac{3}{5}AC,\ \ Cq_3 = \frac{3}{11}AC,\ \ Cq_4 = \frac{2}{5}\cdot AC,$$



the points $q_1 q_2 q_3 q_4$ being respectively on the side of C indicated in the figure. Then $q_1\, q_2\, q_3\, q_4$ are the geometrical foci of the pencil, when first refracted, once reflected, twice reflected, and emergent after two internal reflexions respectively.

260. Suppose a system of parallel rays incident on a refracting sphere—as a rain-drop—and emergent after one internal reflexion.



Let *pqrse* be the course of one of these rays incident parallel to the diameter *PACI*, and passing in a plane which contains C the centre of the sphere, and let us for the present confine our attention to rays which pass in this plane.

Let $\phi\ \phi'$ be the angles of incidence and refraction at q, $\theta =$ arc Ir, i.e. the angle which Ir subtends at C, $\psi = As$; produce *es* backward to meet *pq* produced in t, then $\angle\, etv = D =$ deviation of the ray *pq*.

Now the deviation at each of the refractions at q and s is $\phi - \phi'$, and at the reflexion at r is $\pi - 2\phi'$, and these deviations are all in the same direction;

$$\therefore\ D = 2(\phi - \phi') + \pi - 2\phi' = \pi + 2(\phi - 2\phi') \text{ .... (i).}$$

In order to examine whether D admits of a maximum or minimum for different values of ϕ, we have,

remembering that $\sin\phi = \mu \sin\phi'$, and $\therefore\ \dfrac{d\phi'}{d\phi} = \dfrac{\cos\phi}{\mu\cos\phi'}$,

$$\frac{dD}{d\phi} = 2\left(1 - 2\frac{d\phi'}{d\phi}\right) = 2\left(1 - \frac{2\cos\phi}{\mu\cos\phi'}\right),$$

$$\text{and } \frac{d^2D}{d\phi^2} = \frac{4\sin\phi}{\mu\cos\phi'}\left\{1 - \left(\frac{\cos\phi}{\mu\cos\phi'}\right)^2\right\}.$$

Let ϕ_1 be the value of ϕ which makes $\dfrac{dD}{d\phi} = 0$,

$$\text{then}\quad 2\cos\phi_1 = \mu\cos\phi_1',$$

$$\text{and}\quad \sin\phi_1 = \mu\sin\phi_1';$$

$$\text{whence}\quad \sin\phi_1 = \sqrt{\left(\frac{4-\mu^2}{3}\right)}.$$

Now $\cos\phi'$ is $> \cos\phi$, and $\therefore > \dfrac{\cos\phi}{\mu}$;

and $\therefore\ \dfrac{d^2D}{d\phi^2}$ is positive;

i.e. D is a minimum when $\phi = \phi_1$, and there is only this one value of ϕ for rays incident on the same side of PA which makes D a minimum.

Hence, considering rays incident parallel to PA, as ϕ increases from 0 up to $\dfrac{\pi}{2}$, the corresponding deviation diminishes from π, when $\phi = 0$

$$\text{to } \pi - 2(2\phi_1' - \phi_1) \text{ when } \phi_1 = \sin^{-1}\sqrt{\left(\frac{4-\mu^2}{3}\right)},$$

and afterwards increases from this value up to $2\pi - 4\sin^{-1}\dfrac{1}{\mu}$, as ϕ increases from ϕ_1 up to $\dfrac{\pi}{2}$.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

or $\mu \cos \phi' > < 2 \cos \phi$,

i.e. according as ϕ is $> < \phi_{,}$.

If $\phi = \phi_{,}$, then $v = 4r \cos \phi$, and the primary focus for consecutive refracted rays is on the surface of the sphere for those rays which pass with minimum deviation.

Lastly, $\psi = [illegible] - D$, by (1).

Hence $\frac{d\psi}{d\phi} = 1 - \frac{dD}{d\phi}$, and $\frac{d^2\psi}{d\phi^2} = -\frac{d^2 D}{d\phi^2}$;

if $\phi_{,,}$ be the value of ϕ which makes $\frac{d\psi}{d\phi} = 0$, we obtain,

$$\mu \cos \phi'_{,,} = 4 \cos \phi_{,,},$$

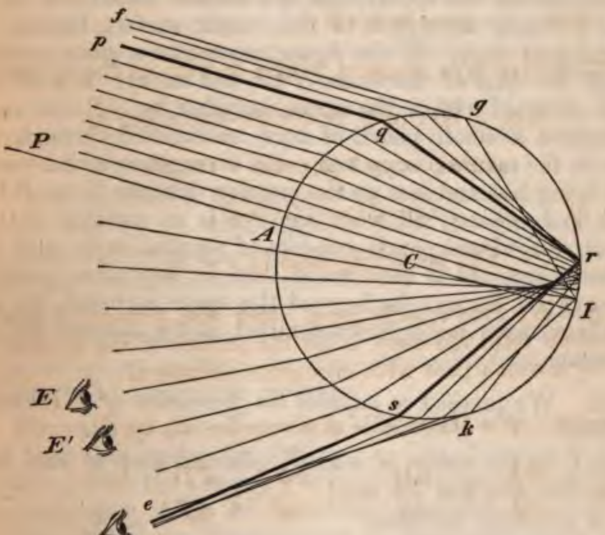
whence $$\sin \phi_{,,} = \sqrt{\left(\frac{16 - \mu^2}{15}\right)},$$

and this value of ϕ renders $\frac{d^2\psi}{d\phi^2}$ negative, or ψ a *maximum*; hence as ϕ increases from 0 up to $\phi_{\prime\prime}$, ψ increases up to a maximum and then diminishes as ϕ goes on increasing from $\phi_{\prime\prime}$ up to $\frac{\pi}{2}$.

It is easily seen that $\phi_{\prime\prime}$ is $> \phi_1$.

261. From the preceding discussion we draw the following inferences.

If *pqrse* be the ray which passes with minimum deviation, then a *small* pencil incident at q has r for its primary focus after refraction, and emerges at s as a pencil of parallel rays in some direction *se*.



As ϕ increases continuously from 0 up to ϕ_1, ψ increases and D diminishes continuously, i. e. the rays incident on the arc Aq emerge from the arc As in a state of *divergence*.

its axis after being refracted into the sphere and once internally reflected, may emerge in a divergent state in the direction tE (Art. 261) and fill the pupil of an eye E, creating the sensation of a bright point in the sky in the direction Et of the species of light which is considered.

Through E draw EB parallel to pq, then since the deviation of pq—and consequently the angle tEB—depends merely on the angle of incidence of pq, and since all rays from the sun may be considered parallel, therefore all drops of rain whose centres lie in a conical surface of which EB is the axis and tEB the semi-vertical angle, will transmit to the eye a similar pencil of divergent rays. A drop of rain at less angular distance from EB than A will transmit to the eye a small pencil with greater deviation and greater divergence than TE, and therefore producing the sensation of a less bright point in the sky,—and the divergence of pencils which reach the eye at a greater angular distance from EB than A is less than that of the pencil tE, until when the deviation is a minimum, the pencil is nearly one of parallel rays, and the impression produced by it on the eye is greater than that of any other pencil.

The impression therefore to the eye produced by pencils transmitted in the way supposed from the drops of rain in a distant shower,—the observer being between the sun and the shower,—would be an illuminated sky of the colour of light, which has been considered, the brightness being greater at greater distances from the line EB, until it is bounded by a circle whose centre is in EB,—beyond which there is no colour, and comparative darkness.

For each species of sun-light this would be the appearance, the bounding circles for different species differing slightly in position, since the amount of minimum deviation depends upon the refractive index of the light (Art. 260). The result of superposing these illuminations of differant colours will be *white light within a certain distance of the line EB terminated by a narrow circular band of vivid prismatic colours arranged in concentric circles about the line EB.* Beyond this band there is no illumination of the sky by pencils transmitted, to the eye in the manner now considered, and it will appear comparatively dark.

Further, considering the finite extent of the sun, the bands of colour such as have been described resulting from pencils proceeding from the several points of the sun, will overlap each other,—the breadth of the band will be increased by the sun's apparent angular diameter, and the colours will be more or less mixed; but the *extreme* colours of the resulting band will be unchanged.

This phenomenon produced by pencils which have been once internally reflected in the rain-drops is called the *Primary Rainbow.*

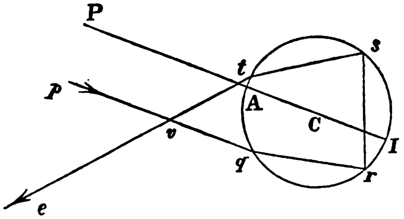
263. Some writers suppose that the pencils received by the eye in a state of divergence,—those namely which emerge through the arc As in Art. 261—are too faint to produce any sensible impression on the eye, and that only those pencils which pass with minimum deviation, and so enter the eye in a state of parallelism, give a sufficiently vivid sensation. This supposition affords the same explanation, as above, of the rings of prismatic colours which constitute the rainbow, but it leaves out of consideration as insensible the faint illumination of which in the previous article we have supposed the bow to be the boundary—which being of a neutral tint is not very striking.

Since, however, no rays can reach the eye from drops *outside* the bow, except such as have been reflected from their surfaces, whilst from drops *within* the bow the eye receives rays which have been internally reflected as well as rays reflected from the external surfaces of the drops, we have an explanation why the space within the bow must appear brighter than the space without it.

The fact that the rain-drops are in falling motion does not interfere with the phenomena,—the place of a falling drop being immediately supplied by another yielding the same kind of pencil of transmitted rays.

264. *Again,* suppose a system of parallel rays incident on a refracting sphere, as a rain-drop, and emergent after *two* internal reflections.

Let $pqrste$ be the course of one of these rays incident parallel to the diameter $PACI$, and passing in a plane which ntains C the centre of the sphere, considering only rays hich pass in this plane.



Let ϕ, ϕ' be the angles of incidence and refraction at q,

$$\psi = \text{arc } IqAt, \quad \theta = Ir, \quad \omega = Is,$$

hen exterior angle $qve = D =$ deviation of the ray pq, and we hall have

$$D = 2(\phi - \phi') + 2(\pi - 2\phi') = 2\pi + 2(\phi - 3\phi');$$

$$\therefore \frac{dD}{d\phi} = 2\left(1 - 3\frac{\cos\phi}{\mu\cos\phi'}\right),$$

$$\frac{d^2D}{d\phi^2} = \frac{6\sin\phi}{\mu\cos\phi'}\left\{1 - \left(\frac{\cos\phi}{\mu\cos\phi'}\right)^2\right\}.$$

If ϕ_1 be the value of ϕ which makes $\dfrac{dD}{d\phi} = 0$,

$$\text{then } 3\cos\phi_1 = \mu\cos\phi_1',$$

$$\text{and } \sin\phi_1 = \mu\sin\phi_1',$$

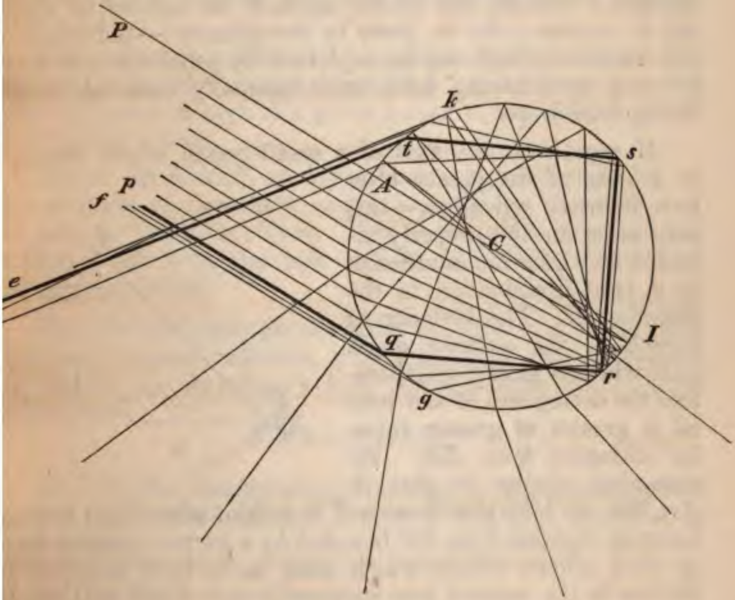
$$\text{whence } \sin\phi_1 = \sqrt{\left(\frac{9-\mu^2}{8}\right)};$$

is value of ϕ renders $\dfrac{d^2D}{d\phi^2}$ positive, and therefore D a *minimum*, and there is only this one value of ϕ, for rays incident n the same side of PA, which makes D a minimum.

long $qr = \frac{3}{4} qr$,—that between the two reflexions the rays are arallel, and that after the second reflexion the primary focus at a distance from t along $ts = \frac{3}{4} ts$.

266. From the preceding discussion we draw the following inferences.

If *pqrste* be the ray which passes with minimum deviaion, then a *small* pencil of which *pq* is the axis, emerges as a encil of parallel rays, of which *te* is the axis. As ϕ increases

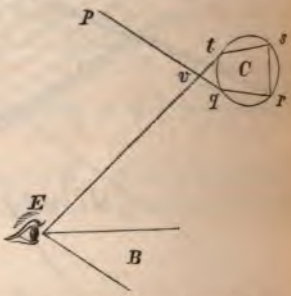


ntinuously from 0 up to ϕ_1, ψ increases and D diminishes ntinuously, i. e. the rays incident on the arc Aq emerge from e arc $IqAt$ in a state of divergence.

If *fg* be the extreme incident ray,—for which $\phi = \frac{\pi}{2}$,—and we take the arc *IqAtk* equal to the maximum value of ψ, then the rays incident on the arc *qg* emerge through the arc *tk* after two internal reflexions, and these emergent rays (as in the case of Art. 261) form by their consecutive intersections a caustic curve of which *te* is an asymptote,—this line being also an asymptote to the *virtual* caustic formed by the rays emergent along *IqAt*, produced backward.

267. *The Secondary Rainbow.*

The space above the primary rainbow, as we have seen (Art. 263), seems darker than the rest; beyond this space appears a broader but fainter rainbow the colours of which are in reverse order to those in the primary:—the origin of this *secondary* bow can be explained by pencils which enter the eye after having been twice internally reflected in the falling rain-drops.



If *pqrstE* be the axis of a small pencil which emerges in a state of divergence after two internal reflexions,—this may enter an eye properly situated and give the impression of a faint illumination in the direction *Et*. The deviation of *pqrstE* is the *exterior* angle *qvE*, which is greater and therefore the divergence of the pencil is greater at greater angular distances from *EB*. By reasoning similar to that in Art. 262, we infer that there will be a faint white light beyond a certain distance from *EB* bounded by a narrow circular band of vivid colours within which there is no light received by the eye in the manner here supposed,—and which will therefore appear comparatively dark. This phenomenon is called the *Secondary Rainbow*.

268. The theoretical existence of rainbows caused by *ncils* which have been three or more times internally re-

ected may be similarly shewn, but the rapid decrease in the ntensity of light in the emergent pencils renders such rainbows with very few exceptions invisible.

In the explanation of the formation of a rainbow the sun was, in the first instance, considered a *point*. To every point f the sun's disc which is of finite extent there will be a orresponding rainbow, as was before noticed, and the visible ainbow resulting from the superposition of these will have ts colours in some degree confused, but its general appearance will be such as has been described.

269. Def. In any rainbow the semi-vertical angle of cone of rain-drops which transmit to the eye pencils of arallel rays of a given colour, is the *radius of the bow* for hat colour.

Def. In any rainbow the angular elevation above the orizon of the highest rain-drop which transmits to the eye a encil of parallel rays of a given colour is the *altitude of the ow* for that colour.

Hence the altitude of any colour in a rainbow added to he altitude of the sun's centre is equal to the radius of he bow for the same colour. If the sun's altitude exceed the adius of the bow, the bow will be below the horizon and herefore invisible.

If D_1 be the minimum deviation of light of a given kind fter one internal reflexion, α the altitude of the corresponding colour in the *primary bow*, A the altitude of the sun's entre, then

$$D_1 + \alpha + A = 180^\circ.$$

If D_1 be the corresponding deviation in the *secondary bow* or the colour whose altitude therein is α, and A the same as efore, then

$$D_1 - \alpha - A = 180^\circ.$$

270. *To investigate the order of the colours in the primary nd secondary rainbow.*

Let the axis of a pencil of sun-light whose refractive ndex is μ be incident on a rain-drop at an angle ϕ and emerge

pound light is considered there will be in addition ncipal bow a series of interior bows decreasing in —to these the name of *spurious* or *supernumerary* ven.

r three of the spurious bows of the primary rainbow times be seen in nature. The results of theory on t have been tested by measurement, by allowing a column of water to run from a vessel, and receiving cope artificial light which has been internally re- his column. (*Cambridge Philosophical Transactions*,

explanation of other optical phenomena in the e, such as Halos, Parhelia, &c. the student may r Young's *Lectures*, or Moigno, *Optique Moderne*,

7. A pencil of parallel rays is incident directly upon a spherical refracting surface, and after refraction converges to a point at a distance from the surface equal to three times the radius—find the index of refraction (i) when the surface is concave, (ii) when it is convex.

8. If a luminous point be seen after reflexion at a plane mirror by an eye in a given position, there is a certain space within which the image of the point can never be situated, however the position of the plane mirror be changed,—find this space.

9. A speck within a solid glass cube is viewed directly through each face in succession; prove that the six images will form the angular points of an octahedron, not generally regular but having all its diagonals equal.

Also compare the volume of the octahedron with that of the cube.

10. A bright point on a glass plate is viewed by an eye close to the plate, and at a given distance from the point; find the direction in which the n^{th} image is seen after successive reflexions within the plate.

11. If the direction of a ray proceeding from a point P on the circumference of a circle, after refraction at the curve pass through a point on the circumference at an angular distance $\frac{\pi}{3}$ from P, find the point of incidence;—$\sqrt{3}$ being the refracting index.

12. A small object is placed in one focus of a prolate spheroid of glass, which is silvered at one of its poles; an eye being placed near the other pole, views the object directly, and by reflexion: if v_1, v_2 be the distances of the images seen from the eye, shew that

$$\frac{1}{v_1}+\frac{1}{v_2}=\frac{4}{\text{latus rectum}}.$$

Shew further that only one image will be seen if $\mu e > 1$,—and supposing $\mu e < 1$ compare the magnitudes of the images.

13. A luminous point is placed at the focus of a prolate spheroid, the index of refraction from which into air is equal

20. At the centre of one end of a glass cylinder a black spot is marked, shew that to an eye at the centre of the other end a number (n) of rings will be visible,—where n is the greatest integer in $\frac{b}{2a\sqrt{(\mu^2-1)}}$, b being the length of the cylinder and a its radius.

21. A luminous point is placed within a reflecting sphere, and the light which falls directly upon the most distant point of the surface is repeatedly reflected. Shew that the distances of the geometrical foci from the centre decrease in harmonic progression.

22. An eye is placed close to the surface of a sphere of glass ($\mu=\frac{3}{2}$) which is silvered at the back; shew that the image which the eye sees of itself is three-fifths of the natural size.

23. A room has its walls covered by mirrors, shew that a man may see himself by reflexion at four of the walls, if he look in a direction parallel to either diagonal.

In what direction must he look in order to see himself after reflexion at three of the walls?

* * * * *

24. A person whose height is h, observes vertically beneath his eye, an object at the bottom of a clear pool: he then removes to a distance d, keeping his eye on the object, when his line of vision makes 45° with the surface; shew that if $\mu^2=2{\cdot}5$, the depth of the pool $=2\,(d-h)$.

25. In one side of a triangle, the interior of which is capable of reflecting light, there are two holes; determine by a construction the position of a point from which rays may enter so as each to pass out after one reflexion.

26. A bright point is moved about an ellipse whose focus coincides with the centre of a reflecting sphere, of such a magnitude that the directrix corresponding to the focus meets the sphere. Prove that the positions for which the geometrical focus of the point is a point in the same ellipse, are those in which the ellipse is cut by the diameters of the sphere which are drawn to the points where the directrix meets it.

[illegible] microscope:—uni
[illegible] of the area of the
[illegible] transverse section of the

33. A luminous point is in the centre of an equilateral iangle; shew by considering the course of a ray parallel to ie side, that the distance of the image from the luminous int for $2n$ reflexions is na, and for $2n+1$ reflexions

$$a\sqrt{(n^2+n+\tfrac{1}{3})},$$

being a side of the triangle.

34. A ray is incident on a refracting sphere; shew that ter one internal reflexion it will emerge parallel and re-ersed, if $\cos\phi'=\frac{\mu}{2}$; and after two internal reflexions parallel ıd proceeding, if $\cos\phi'=\frac{1}{2}\sqrt{(\mu+1)}$, ϕ' being the internal ıgle of incidence.

35. A cylindrical tumbler being placed upon a table, a erson is so situated that looking over the nearest point of ıe edge he can just see half down the inside. When a fluid l is poured into the tumbler he can, by refraction, just see ıe bottom when the tumbler is three-fourths filled:—when fluid B is poured in he cannot see the bottom till the tumbler s quite full. Supposing the refractive indices from air into l and B to be as $\sqrt{2}:1$,—shew that the height of the tumbler s double the breadth of its base.

36. Three plane mirrors are joined together by their dges, so that their planes are at right angles to each other, nd their reflecting surfaces turned inwards. Shew that a man ooking into the solid angle so formed will see his face in-erted; and trace the course of the pencils by which he sees he different parts of his face.

37. A ray of light after reflexion at two plane surfaces, eturns to the point from which it emanated, having in its ourse between the mirrors passed through a given point; if he mirrors be moved in any way consistently with these cir-umstances, shew that their intersection will always lie in one lane; and may be anywhere in that plane, the length of the ay being supposed constant.

38. A rectangular box, at the bottom of which is a plane ıirror, contains an unknown quantity of water; from the

line through the

which the sine

distances of

spherical

geometrical foc

$$\frac{1}{QQ'} = \frac{\mu -}{}$$

plane reflectors, the

that the line joining

plane of each, and a brig

the greatest

looking over

by the greatest

reflecting cir

produced to m

be the radius

the axis of the rad

the inclin

Further,—shew that if from the point of incidence of each ay there be drawn in a direction opposite to that of the eflected ray, a line equal in length to the incident ray, the ocus of the extremities of these lines is a curve cutting the ines at right angles,—and the equation of which referred to he radiant point as origin, and the axis QO as axis of x, is

$$x^2+y^2-2ux=2a\sqrt{(x^2+y^2)}.$$

Shew that the origin is a double point; and trace the urve: shew also that the equation may be expressed in he form

$$\sqrt{\{(x-\alpha)^2+y^2\}}=m\sqrt{(x^2+y^2)}+n.$$

CHAPTER II.

Illumination of Surfaces.

1. Two candles of unequal brightness and height are laced upon a horizontal table,—shew that the locus of the oint which is equally illuminated by the candles is a circle.

2. A circular disc exactly fits a hole in a wall and evolves in its own plane about its centre; $2n$ equidistant radii re drawn on it, and the space between every alternate pair emoved: supposing the image of every object to remain the $^{\text{th}}$ part of a second upon the retina of the eye, find the smallst angular velocity which may be given to the disc consistntly with the eye having an unobstructed view through the ole.

If the disc itself be looked at under these circumstances, hat will be the appearance presented?

3. Why does the apparent brightness of a light seen by ight—(neglecting such disturbing causes as absorption by he air)—remain the same at all distances?

4. A candle is placed at a given distance in front of a ertical plane circular mirror on a line perpendicular to the orizontal diameter at its extremity; shew that the boundary f the reflected light, which falls on a wall of which the plane

is perpendicular to that of the mirror, is a parabola;—and determine its latus rectum.

5. A luminous globe falls from a point above the Earth's surface in a dark night: shew that it will look like a bright falling column, elongating as it descends.

If c_1, c_2, c_3 be the lengths of the apparent column at the ends of times t_1, t_2, t_3 from the commencement of the fall,—prove that, gravity being considered constant, and the resistance of the air being neglected,

$$t_1(c_2 - c_3) + t_2(c_3 - c_1) + t_3(c_1 - c_2) = 0.$$

6. A thin plate of glass is placed parallel to a screen, and there is a luminous point in any given position on the other side of it; the glass is cracked in one spot in the shape of two straight lines forming a cross, the planes of the cracks being perpendicular to that of the glass: shew that there will be two corresponding figures of a cross thrown upon the screen, the one bright and the other dark: shew also that the luminous point may move in a certain fixed line without producing motion in the figures.

7. A stick of given length is suspended vertically in a room. A candle is carried past the stick in such a way that the shadow of the stick falls partly on one wall, and partly on the ceiling,—the extremities of the shadow on each lying in two straight lines: find the locus of the candle.

8. An opaque sphere is placed upon a plane, and in the diameter passing through the point of contact a luminous point is placed,—its distance from the sphere being equal to the radius,—

Prove that the area of the shadow cast on the plane is three times that of a great circle of the sphere.

If a transparent liquid, whose refractive index is $\sqrt{3}$ be placed above the plane, so as just to cover the sphere, shew that the area of the shadow will then be reduced to twice that of the great circle.

9. A plane drawn through a given point is illuminated

o self-luminous spheres; find the position of the plane the illumination at the given point is a maximum.

. If a candle be placed upon a table at a given dis- from a point in that table, what must be the height of ndle that the brightness at that point may be the great- ssible?

. The shadow cast by an oblate spheroid resting on rtex—on a horizontal plane, in the sun,—is an ellipse; e spheroid stands in its focus.

* * * * *

. Two semicircular self-luminous plates are placed heir diameters upon a plane, and with their planes per- ular to the line which joins their centres A and B; find sition of that point on the line AB where the illumi- of the plane upon which the semicircles are placed is

. If a concave mirror revolve round an axis through tre, slightly inclined to the axis of the mirror, find the ance on a screen placed at the farther focus, when a of electric sparks passes in quick succession through arer focus in the axis of revolution.

. A small plane touches a self-luminous paraboloid of tion at its vertex and is then moved parallel to itself the axis produced; prove that the illumination of the varies inversely as its distance from the focus.

. Three candles are placed in a room, and the two r being lighted throw shadows of the third upon the ; if the directions of these shadows be produced, where ey meet?

. If a globe be placed upon a table, the breadth of the shadow cast by a candle (considered as a luminous will be independent of the position of the globe.

. The centre of a spherical ball is moveable in the l plane which is equidistant from two candles on a

16—2

30. If c_1, c_2, c_3 be the lengths of the meridian shadows of three equal vertical gnomons, on the same day, at three different places on the same meridian,—prove that the latitudes λ_1, λ_2, λ_3 of the places are connected together by the equation

$$c_1 \cdot \frac{(c_2-c_3)^2}{\tan(\lambda_2-\lambda_3)} + c_2 \cdot \frac{(c_3-c_1)^2}{\tan(\lambda_3-\lambda_1)} + c_3 \cdot \frac{(c_1-c_2)^2}{\tan(\lambda_1-\lambda_2)} = 0.$$

CHAPTER III.

Aberrations of small direct pencils.

1. A pencil of parallel rays is incident on a convex reflecting surface—y, r being the semi-aperture and radius of the mirror,—the diameter of the least circle of aberration $= \dfrac{y^3}{4r^2}$.

2. A small pencil is directly refracted at a plane surface,—the radius of the least circle of aberration $= \dfrac{\mu^2-1}{8\mu^2} \cdot \dfrac{y^3}{u^2}$.

3. A pencil of parallel rays is refracted directly at a spherical surface,—the radius of the least circle of aberration $= \dfrac{y^3}{8\mu^2 r^2}$.

4. A mirror of given aperture and focal length, and of small curvature, has the form of a prolate spheroid;—shew that the aberration for parallel rays varies inversely as the major axis.

5. A segment of a paraboloid of revolution is cut off by a plane perpendicular to the axis so as to form a plano-convex lens: when a pencil of rays parallel to the axis is refracted through the lens, find the aberration in terms of the thickness,—which is so small that its square may be neglected.

6. If the law of refraction were assumed to be $\phi = \mu\phi'$, *shew that* the approximate error in finding the point where

ay incident near and parallel to the axis of a spherical racting surface cuts the axis after refraction, would be $-\frac{\mu+1}{\mu-1}\cdot\frac{y^2}{r}$,—using the ordinary notation.

CHAPTER IV.

Focal lines of small oblique pencils.

1. The distance between the focal lines of a small oblique pencil after refraction at a plane surface is $=\left(\mu-\frac{1}{\mu}\right)\tan^2\phi\,.\,u$.

2. A small pencil of parallel rays enters a reflecting sphere, and after n internal reflexions emerges through the same aperture; find the angle of incidence and the position of the primary focal line at emergence.

3. An eye is placed under and close to the surface of a clear and stagnant fluid, and a vertical straight rod of given length is placed at a given distance from the eye; determine the form of the image, and the altitude of its highest point above the surface of the fluid.

4. If an eye be placed in air close to the surface of a clear stagnant fluid, shew that the apparent form of a circular arc in the fluid, whose centre coincides with the place of the eye, and whose plane is perpendicular to the surface, is defined by the equation

$$r=a\,\frac{\mu\sin^2\theta}{\mu^2-\cos^2\theta},$$

where $a=$ radius of circle, $\mu=$ refractive index, and the radius vector r drawn from the position of the eye makes the angle θ with the surface.

5. A small pencil of homogeneous light is incident obliquely on a plane refracting surface; shew that if the obliquity be small, the distance of the point of incidence

[illegible] impossible [illegible]

[illegible] conditions [illegible] that the primary [illegible] in the interior [illegible] of a small pencil [illegible] at an infinite distance [illegible] that this is only [illegible] from the centre is greater than [illegible]

[illegible] Prove the following geometrical con[illegible] position of the primary focal [illegible] for

(i) after oblique reflexion at a spherical surface, (fig. rt. 66), Q the origin, O, A the centres of the surface and of e face of the reflector, Aq_1 the direction of the ray after flexion.

Draw OF perpendicular to QA, FG perpendicular to O, join QG and produce it to meet Aq_1 in q_1 which will be e primary focus.

(ii) after refraction at a spherical surface, (fig. Art. 68), the origin, Aq_1 the refracted ray.

Draw QF, FG perpendicular to AQ, AO respectively:—oin GO meeting Aq_1 in K,—draw KV, Vq_1 perpendicular to O, Aq_1 respectively,—then q_1 is the primary focus.

(iii) Adapt the case (ii) to the case of oblique refraction t a plane surface.

12. A small pencil of rays suffers a series of reflexions ithin a polished spherical surface; if the rays are initially arallel, the primary focus after the n^{th} reflexion divides the hord of the surface which is the axis of the pencil in the atio $2n-1 : 2n+1$.

Caustics.

13. Rays parallel to the axis of y are incident on the eflecting curve $y=\epsilon^x$, the catacaustic is the catenary

$$y=\tfrac{1}{2}\{\epsilon^{x+1}+\epsilon^{-(x+1)}\}.$$

14. Rays are incident on the parabola $y^2=4ax$ perpendiular to its axis,—the equation to the catacaustic is

$$y^2=\left(\frac{9a-x}{3a}\right)^2\cdot\frac{ax}{3}.$$

Discuss the form, &c., of this curve.

15. Rays are incident on a circle from a point S within he circumference, and CO is drawn perpendicularly from the entre C on any ray SQ reflected at Q to the point P of the caustic; prove that the locus of P may be traced by means of the equation

$$\frac{1}{\sqrt{(CP^2-CO^2)}}=\frac{1}{SO}+\frac{2}{QO}.$$

16. Rays emanating from a point S are reflected at a plane curve—if SY be the perpendicular from S on the tangent to the curve at any point P, and SY be produced to a point Q such that $QY=SY$, then will the caustic curve be the evolute of the locus of Q.

The same problem may be stated thus:

If with each point successively of the reflecting curve as centre, and its distance from the radiant point as radius, we describe a series of circles,—the envelope of all these circles will be a curve, the evolute of which will be the caustic.

17. Assuming the formula for the position of the primary focal line in reflexion at a spherical surface,—prove that if Q be the radiant point within the circle, QP the ray corresponding to an asymptote of the caustic, and P' the point in which PQ produced meets the circle, then $QP'=3.QP$: and hence deduce a geometrical construction for the asymptotes of the caustic.

18. A radiant point being placed in front of a looking-glass, find the caustic curve formed by the rays emergent from the glass after reflexion at the quicksilver.

19. Parallel rays are incident upon a cylinder in a direction perpendicular to its axis; shew that the equation to a section of the caustic surface is

$$(\mu^2-1)\,y=\{(\mu^2a)^{\frac{2}{3}}-x^{\frac{2}{3}}\}^{\frac{3}{2}}+\{a^{\frac{2}{3}}-(\mu^2x)^{\frac{2}{3}}\}^{\frac{3}{2}},$$

the centre of the corresponding section of the cylinder, whose radius is a, being the origin, and its diameter perpendicular to the incident rays, the axis of x,—μ the refractive index.

20. Rays proceeding from a luminous point in the pole of the spiral $r=c\epsilon^{\theta\cot\alpha}$, are reflected at the curve, the cata-*caustic* is a similar spiral. *Further*, the spiral will be its

own catacaustic if $\frac{1}{2}\sec\alpha = \epsilon^{(2n\pi-\alpha)\cot\alpha}$, where n = any positive integer.

21. Rays fall on the cycloid $y = a \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{(2ax - x^2)}$ parallel to the axis of x, the catacaustic is two cycloids of half the dimensions of the original given by the equation

$$\pm y \pm \frac{a\pi}{2} = \sqrt{\{(2a-x)(x-a)\}} + a\cos^{-1}\sqrt{\left(\frac{x-a}{a}\right)}.$$

22. If $\alpha\beta$ be the co-ordinates of a radiant point, the caustic curve formed by reflexion at the circle $x^2 + y^2 = c^2$ is

$$[4(\alpha^2+\beta^2)(x^2+y^2) - c^2\{(x+\alpha)^2 + (y+\beta)^2\}]^3$$
$$= 27c^4(\beta x - \alpha y)^2(x^2+y^2-\alpha^2-\beta^2)^2.$$

Cambridge and Dublin Math. Journal, II. pp. 128, 236.

The student may refer to several papers on the Caustics by reflexion at a circle, by Mr Holditch, in the *Quarterly Journal of Mathematics*.

CHAPTER V.

Successive reflexions and refractions—prisms—lenses.

1. Draw a line between two plane mirrors which may be the direction of a ray proceeding from a given luminous point after reflexion at either of the mirrors.

2. When a ray of light is reflected at any surface, the incident and reflected rays are equally inclined to any plane which passes through the normal at the point of incidence.

3. What is the greatest apparent zenith distance which a star can have as seen by an eye under water?

4. A pencil passes directly from air into water, through a plate of glass, to find its geometrical focus, assuming

$${}_A\mu_G = \frac{3}{2}, \quad {}_A\mu_W = \frac{4}{3}.$$

5. What must be the inclination of two mirrors in order that a ray incident parallel to one of them may, after two reflexions, be parallel to the other.

6. A ray of light passes from one medium through three others bounded by parallel planes, the refractive indices taken in order being $\sqrt{3}$, $\sqrt{6}$, $\sqrt{2}$, $\sqrt{6}$;—if the angle of incidence at the first refraction be $\frac{\pi}{4}$, find the direction of the ray in each of the plates.

7. If the air near the surface of the ground be less dense than at small altitudes above it, there will be observed inverted images of distant horizontal objects.

Hence explain the *Mirage*.

8. If rays are incident upon a common looking-glass in an oblique direction from a candle,—*one* faint image is observed before the *principal* image,—and a row behind it diminishing rapidly in brightness. Explain this.

9. If a transparent plate lie between two media of unequal densities each less than that of the plate, and if a ray, passing from the denser of the external media into the plate, fall at the critical angle on the third medium,—the angle of incidence at the first surface of the plate will be the critical angle from the first on the third medium.

10. A ray passing through a point Q is incident upon a refracting plate; q is the intersection of the emergent ray, produced backwards, with the normal to the plate through Q; if the angle of incidence be $=\tan^{-1}\mu$ and t the thickness of the plate, prove that

$$Qq = \frac{\mu^2 - 1}{\mu^2} \,.\, t.$$

11. A rod, inclined at any angle to a plate of glass, is seen by an eye on the opposite side of the plate; shew that the length of the image of the rod formed by geometrical foci is equal to the length of the rod.

Is the image formed by the refraction at the first surface *of the same* magnitude as either?

12. A glass plate of sensible thickness is in contact with the surface of still water, find how much the image of a point in the water will be elevated.

Find also the area of the upper surface of the glass which is effective in transmitting rays.

* * * * *

13. A luminous point is seen through a triangular glass prism, by means of pencils once internally reflected, trace the course of the axis of a pencil from the luminous point to the eye,—$\frac{3}{2}$ being the index of refraction from air into glass.

14. A prism is held before the eye with the edge vertical, and one face perpendicular to the axis of the eye; how much in azimuth of the horizon will the observer be prevented from seeing by the interposition of the prism?

15. A small pencil diverging from a given point is refracted through a prism in a principal plane and very near the edge—if the prism be turned about its edge, form the equations necessary for determining the curves on which the foci lie, and shew that these curves intersect.

16. Rays are incident at one point of a prism in all directions in a principal plane,—shew that if the refracting angle be $> \sin^{-1}\frac{1}{\mu}$, rays incident from that side of the normal which is towards the edge of the prism will not pass through,—and examine what rays will pass through.

17. If there be a small speck upon the middle of one of the sides of an equilateral prism, and a person place his eye close to the opposite edge in the plane drawn through the speck perpendicular to the axis of the prism: shew that he will see *two* specks, and find the apparent angular distance between them.

18. A small pencil of rays is transmitted through a prism in a principal plane—find the focus of emergent rays in the primary plane, taking account of the thickness of the prism;—and deduce from the result the expression for the

place of the focus when a pencil is transmitted through a plate which is bounded by parallel plane surfaces.

19. Two equal prisms of the same refracting substance are placed in contact: shew that a ray of given species will be just incapable of transmission through the compound prism thus formed when the mutual inclination of their edges is $2\sin^{-1}\left(\frac{1}{\mu\sin\alpha}\right)$, α being the refracting angle of each prism.

20. Can objects be seen—(without internal reflexion)—through a prism whose refractive index is $\frac{3}{2}$ and refracting angle a right angle?

21. Two prisms are in contact with their edges perpendicular to one another; a ray perpendicular to the edge of one prism being refracted through the combination emerges from the second perpendicular to its edge: determine the deviation, and shew that the problem is impossible if the angle of either prism exceed the critical angle.

22. The deviation of a ray refracted through a prism of small angle in a principal plane

$$=\left(\mu\frac{\cos\phi'}{\cos\phi}-1\right)i, \text{ nearly,}$$

ϕ, ϕ' being the angles of incidence and refraction at the first surface. In what case will the deviation be $=(\mu-1)\,i$, nearly?

23. If a ray of light $QACS$ be refracted through a prism IKL in a principal plane, and if the vertical angle $KIL=\alpha$, $QAK=\theta$, $ACL=\phi$, and the whole deviation of the ray $=\delta$, then will

$$\tan\left(\phi-\frac{\alpha}{2}\right)\tan\frac{\alpha}{2}=\tan\left(\theta+\frac{\delta+\alpha}{2}\right)\tan\frac{\delta+\alpha}{2}.$$

24. A small pencil of rays traverses a prism in a principal plane with minimum deviation. The length of the path of the axis within the prism being c, and i the angle of the prism,—shew that the distance between the primary and secondary foci after emergence $=\frac{\mu^2-1}{\mu}\left(\tan\frac{i}{2}\right)^2 c$.

25. A ray passes through a prism in a principal plane, the deviation being equal to the angle of incidence, and each of them equal to twice the angle of the prism,—shew that the latter angle $=\cos^{-1}\sqrt{\left(\frac{\mu^2-1}{8}\right)}$.

26. When a ray is refracted through a prism in a principal plane, prove with the notation of Art. (91) that

$$\sin(D+i-\phi)=2\mu\sin\frac{i}{2}\cos\left(\frac{i}{2}-\phi'\right)-\sin\phi.$$

* * * * *

27. A glass lens $\left(\mu=\frac{3}{2}\right)$ of given focal length is placed under water $\left(\mu=\frac{4}{3}\right)$; find the geometrical focus of a pencil of parallel rays directly incident upon the lens.

28. The least distance between an object and its image formed by a plano-convex lens of glass is 12 inches, find the radius of the spherical surface $\left(\mu=\frac{3}{2}\right)$.

29. A person views an object through a convex lens with both eyes; trace the pencil by which each eye sees the object.

30. A convex lens, held 12 inches from a wall, forms on the wall a distinct image of a candle; when the lens is held 6 inches from the wall, it is found that, to produce a distinct image of the candle on the wall, the distance of the candle from the lens must be doubled;—find the focal length of the lens.

31. From a prolate spheroid, formed of glass, portions are removed by planes passing through the two foci and perpendicular to the axis; if a luminous point be situated at one of the foci, and the eye placed close to the other, describe the appearance which will be presented, so far as it is due to pencils which have suffered one internal reflexion.

Rays issuing from a luminous point in its a on a thin lens. A portion of those that en wed to proceed at once through the seco ond portion however does not escape unti been twice internally reflected; a third portion fou reflected, and so on. Shew that a row of images formed whose distances from the lens are in harmo gression.

33. In the annular lens of a Lighthouse,—investi relation between its focal length and the radius of c of a section of one of its concentric rings by a plane its axis.

34. If a ray of light be incident upon a refracting and the directions of the incident and emergent rays p meet the surface in the same point; find the angle of i when $\mu=\frac{3+\sqrt{5}}{2}$.

35. A sphere composed of two hemispheres of refractive powers is placed in the path of a pencil of such a manner that the axis of the pencil is perpendi the plane of junction and passes through the centre; de the geometrical focus of the refracted pencil.

36. Find the geometrical focus of a pencil of fracted through a hollow glass sphere, whose exter internal radii are r, r' respectively.

37. Shew what kind of lens will be required to e person to see distinctly under water.

If an eye be under water, shew how to determ course of the ray by which any object above the seen.

38. A plano-convex and a plano-concave lens power are fitted together so as to form a plate;—if th be slightly separated, remaining co-axial, examine the they will produce in the divergency of a small pencil gent rays incident directly (i) on the negative lens, the positive lens.

39. Two plano-convex lenses of the same size and form are placed so as to make a double-convex lens: find the focal length of this lens, the refractive indices of the two substances being μ and μ'.

40. A convex lens is moved between a candle and a vertical screen, the distance of which is greater than four times the focal length of the lens: prove that two real images of the candle will be formed on the screen:—and if the ratio of their magnitudes is m, find the focal length of the lens.

41. The focal length of a convex lens in air is 15 inches. When its lower surface is immersed in a certain fluid, a vertical pencil of parallel rays is brought to a focus at a depth of 24 inches: when the other surface is immersed, the depth of the focus is 40 inches; and when the lens is wholly immersed the focal length is 60 inches. Find the refractive indices of the lens and fluid, and the radii of the surfaces of the lens.

42. A pencil of rays is directly incident on a double concave lens the second surface of which is silvered; find the geometrical focus of the pencil after emergence, and the form of the spherical surface which, placed at the same distance as the lens, would reflect rays to the same focus.

43. If the radius of the anterior surface of a concave glass speculum of inconsiderable thickness $(c) = a$, and if he radius of the second surface $= a + \frac{13c}{9}$, the image of a disant object formed by reflexion at the first surface will coinide with the image formed by reflexion at the second surface, and by refraction at the first.

44. A pencil of rays is directly refracted through a series of lenses separated by finite intervals $a_1, a_2, \ldots a_{n-1}$, the axes being coincident. Shew that if $\frac{1}{k_1}, \frac{1}{k_2} \ldots \frac{1}{k_n}$ be the focal

[illegible] issuing from a luminous point in its axis [illegible] thin lens. A portion of these [illegible] to proceed at once through the [illegible] [illegible] portion however does not emerge until it has been twice internally reflected; a third portion four times reflected, and so on. Shew that a row of images will be formed whose distances from the lens are in harmonic progression.

33. In the annular lens of a Lighthouse,—investigate the relation between its focal length and the radius of curvature of a section of one of its concentric rings by a plane through its axis.

34. If a ray of light be incident upon a refracting sphere and the directions of the incident and emergent rays produced meet the surface in the same point; find the angle of incidence when $\mu = \frac{3+\sqrt{5}}{2}$.

35. A sphere composed of two hemispheres of different refractive powers is placed in the path of a pencil of light in such a manner that the axis of the pencil is perpendicular to the plane of junction and passes through the centre; determine the geometrical focus of the refracted pencil.

36. Find the geometrical focus of a pencil of rays refracted through a hollow glass sphere, whose external and internal radii are r, r' respectively.

37. Shew what kind of lens will be required to enable a person to see distinctly under water.

If an eye be under water, shew how to determine the course of the ray by which any object above the water is seen.

38. A plano-convex and a plano-concave lens of equal power are fitted together so as to form a plate;—if the lenses be slightly separated, remaining co-axial, examine the change they will produce in the divergency of a small pencil of divergent rays incident directly (i) on the negative lens, (ii) on the positive lens.

39. Two plano-convex lenses of the same size and form are placed so as to make a double-convex lens: find the focal length of this lens, the refractive indices of the two substances being μ and μ'.

40. A convex lens is moved between a candle and a vertical screen, the distance of which is greater than four times the focal length of the lens: prove that two real images of the candle will be formed on the screen:—and if the ratio of their magnitudes is m, find the focal length of the lens.

41. The focal length of a convex lens in air is 10 inches. When its lower surface is immersed in a certain fluid, a vertical pencil of parallel rays is brought to a focus at a depth of 24 inches: when the other surface is immersed, the depth of the focus is 40 inches; and when the lens is wholly immersed the focal length is 60 inches. Find the refractive indices of the lens and fluid, and the radii of the surfaces of the lens.

42. A pencil of rays is directly incident on a double concave lens the second surface of which is silvered; find the geometrical focus of the pencil after emergence, and the form of the spherical surface which, placed at the same distance as the lens, would reflect rays to the same focus.

43. If the radius of the anterior surface of a concave glass speculum of inconsiderable thickness $(c) = a$, and if the radius of the second surface $= a + \frac{13c}{9}$, the image of a distant object formed by reflexion at the first surface will coincide with the image formed by reflexion at the second surface and by refraction at the first.

44. A pencil of rays is directly refracted through a series of lenses separated by finite intervals $a_1, a_2, \ldots a_{n-1}$, the axes being coincident. Shew that if $\frac{1}{k_1}, \frac{1}{k_2} \ldots \frac{1}{k_n}$ be the focal

lengths of the lenses, the geometrical focus will be given by the equation

$$v_n = \frac{1}{k_n +}\,\frac{1}{a_{n-1} +}\,\frac{1}{k_{n-1} +}\cdots\frac{1}{k_1 +}\,\frac{1}{u}.$$

45. $ABCD$ is a double convex lens (axis BD) whose thickness $t = \mu$ (*sum of radii of surfaces*). P is a dark spot within the lens at the point of intersection of AC and BD; Q, R the images of P seen from two points in the axis on different sides of the lens. Then, if f be the focal length of a thin lens which has the same radii and refractive index as the lens $ABCD$,

$$QR = \frac{4f}{\mu - 2} + \frac{\mu^2 - 3\mu + 3}{(\mu - 1)(\mu - 2)}\,t.$$

46. The distances of the conjugate foci from the first and second focal centres of a lens are connected by the equation

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s} - \frac{\mu - 1}{\mu}\cdot\frac{t}{rs}\right).$$

CHAPTER VI.

Refraction through media of varying density.

1. The index of refraction at any point of a medium bounded by a plane is $\epsilon^{\frac{x}{c}}$, where x is the distance of the point from the plane; find the equation to the path of a ray incident at an angle α on the bounding plane, and shew that it will have an asymptote perpendicular to the plane at a distance $c\alpha$ from the point of incidence.

2. The index of refraction (μ) in a medium varies from point to point, being a function of the distances x and y from two planes at right angles to each other; a ray traverses the medium in a plane perpendicular to these two planes; if

log $\mu =$ [illegible]
may value [illegible]

[illegible]
bound [illegible]
its first [illegible]
dium [illegible]
posed [illegible]
medi [illegible]

[illegible]
shew that [illegible]
there [illegible]
out the [illegible]
exception [illegible]

[illegible]
refraction [illegible]

ie $= \mu$ [illegible]
whose [illegible]
directly [illegible]
incident [illegible]

[illegible]
small [illegible]
at any [illegible]
and [illegible]
that [illegible]
pendicular [illegible]

[illegible]
the [illegible]
ray [illegible] to the axes [illegible]
distance [illegible] shew that [illegible]

be [illegible] measure [illegible]

8. A ray of light passes through a medium bounded by parallel planes, the density of which varies in such a manner that the index of refraction at any point $= 1 + kx$, where x is the distance of the point from the plane on which the ray is first incident. The angle of incidence being α and the point of incidence the origin, shew that the path of the ray is defined by the equation

$$\cos^2\frac{\alpha}{2} \cdot e^{\frac{ky}{\sin\alpha}} + \sin^2\frac{\alpha}{2} \cdot e^{\frac{-ky}{\sin\alpha}} = 1 + kx.$$

9. A vessel of depth a, the p and bottom of which are horizontal planes, is filled with a transparent fluid, the refractive index of which at a depth z below the surface is $1 + \frac{z}{a}$. Two small holes being made in the top, a ray of light enters at one hole, is reflected at the bottom and emerges at the second hole; shew that the dist e between the holes must not be $> 2a \log_e (2 + \sqrt{3})$.

10. Shew that if $\mu^2 - 1 = k\rho$ for any state of density of the Earth's atmosphere, the following differential equation of the trajectory of a ray of light coming from a heavenly body results independently of any *Theory of Light*, viz.:

$$d\theta = \frac{a \sin z \, . \, dr}{r\sqrt{(r^2 - a^2 \sin^2 z + kr^2\rho)}},$$

z being the apparent zenith distance of the body at the Earth's surface, a the Earth's radius, and r, θ polar co-ordinates referred to the Earth's centre and to the radius drawn to the place of observation.

11. The refractive index at any point of a spherical shell is a function $f(r)$ of the distance r from the centre,—and a pencil of light, proceeding from a point at a distance c from the centre is directly refracted through the shell; shew that the distance (c') from the centre of the geometrical focus after emergence is given by the equation

$$\tfrac{1}{2}\{f(a)\}^2\left(\frac{1}{c'} - \frac{1}{c}\right) = \frac{f(b)\{f(b) - 1\}}{b} - \frac{f(a)\{f(a) - 1\}}{a} - \int_b^a \frac{f'(r)}{r} . dr,$$

here a, b are respectively the external and internal radii of
ie shell, and c' is measured in the same direction as c.

Reflexion, &c. in any manner.

12. A ray of light whose direction-cosines referred to
ree rectangular axes are λ, μ, ν is incident at an angle ϕ on
reflecting plane

$$lx + my + nz = 0;$$

ew that the direction-cosines of the reflected ray are equal to

$$2l \cos\phi - \lambda, \quad 2m \cos\phi - \mu, \quad 2n \cos\phi - \nu.$$

13. A ray is reflected at two plane mirrors so that the
lanes of reflexion are at right angles; shew that the devia-
ion is a minimum when the angles of incidence are equal.
nd if ϕ be the angle of incidence, D the minimum deviation,
the inclination of the mirrors, shew that

$$\cos i = \cos^2\phi, \text{ and } \cos D = \cos^2 2\phi.$$

14. Prove that when a ray of light is reflected at any
urve-surface the length of the course of the ray, reckoned
rom any point in the incident ray to any point in the reflected,
s a minimum.

Camb. and Dub. Math. Journal, Vol. II. p. 288.

If a polished plane have an indefinite number of very fine
oncentric circular grooves turned on its surface, and light be
ncident on it from a luminous point, the appearance presented
o the eye of an observer will be that of a bright curve; find
ts equation.

15. If a ray of light be reflected once at each of two
lane surfaces in any planes, shew that

$$\cos D = \cos 2I + 2(\sin I \sin\phi \sin\theta)^2,$$

here D is the whole deviation of the ray, I the angle be-
ween normals to the surfaces, ϕ the angle of incidence of the

first incident ray, and θ the angle between the plane of incidence of the first incident ray and a plane perpendicular to the intersection of the reflecting surfaces.

10. There are three plane reflectors, two of which are at right angles to each other, and a ray of light is incident upon the third and reflected successively by each of them;—it is required to show that the angle between the first incident and last reflected rays is equal to twice the angle of incidence upon the first surface.

11. An oblique cone of rays falls upon a narrow reflecting annulus cut from the surface of a cone by planes perpendicular to the axis, find the general form of the reflected image thrown on screens at different distances perpendicular to the axis of the cone.

12. A ray of light is incident parallel to the axis of [illegible] whose equation is

[illegible]

[illegible] of the reflect[illegible]

[illegible]

[illegible] parabo[illegible]

[illegible] surface $x =$ [illegible]

[illegible] angle [illegible]

[illegible]

[illegible]

the [illegible]

[illegible]

first incident ray, and θ the angle between the plane of incidence of the first incident ray and a plane perpendicular to the intersection of the reflecting surfaces.

16. There are three plane reflectors, two of which are at right angles to each other, and a ray of light is incident upon the third and reflected successively by each of them;—it is required to shew that the angle between the first incident and last reflected rays is equal to twice the angle of incidence upon the first surface.

17. An oblique cone of rays falls upon a narrow reflecting annulus cut from the surface of a cone by planes perpendicular to the axis; find th eral form of the reflected image thrown on screens at ent distances perpendicular to the axis of the cone.

18. A ray of light is incident parallel to the axis of a reflecting elliptic paraboloid whose equation is

$$\frac{y^2}{b} + \frac{z^2}{c} = 2x$$

at a point (α, β, γ); shew that the equations of the reflected ray are

$$\frac{x-\alpha}{\left(\frac{\beta}{b}\right)^2 + \left(\frac{\gamma}{c}\right)^2 - 1} = \frac{y-\beta}{\frac{2\beta}{b}} = \frac{z-\gamma}{\frac{2\gamma}{c}}.$$

Each reflected ray will pass through each of two parabolas lying in the principal planes of the paraboloid.

19. A ray is incident at a point x, y, z of a surface $u=0$, and there is both a reflected (i) and refracted (ii) ray,—if l, m, n be the direction-cosines of the normal to the surface $u=0$ at x, y, z,—$\alpha\beta\gamma$ those of the incident ray, i the angle of incidence and reflexion, i' that of refraction—the equations to the reflected and refracted rays are severally

$$\frac{X-x}{2l\cos i-\alpha} = \frac{Y-y}{2m\cos i-\beta} = \frac{Z-z}{2n\cos i-\gamma} \ldots\ldots\ldots\ldots\text{(i)},$$

$$\frac{X-x}{-i')+\alpha\sin i'}=\frac{Y-y}{m\sin(i-i')+\beta\sin i'}=\frac{Z-z}{n\sin(i-i')+\gamma\sin i'}\ \text{(ii)}.$$

). A ray of light is incident from the centre of an ellip-
the inner surface of which is polished, and the equation
ich is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0;$$

that the equations to the reflected ray will be

$$\frac{X-x}{x\left(2\frac{p^2}{a^2}-1\right)}=\frac{Y-y}{y\left(2\frac{p^2}{b^2}-1\right)}=\frac{Z-z}{z\left(2\frac{p^2}{c^2}-1\right)},$$

x, y, z are the co-ordinates of the point of incidence,

$$\frac{1}{p^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}.$$

ırther,—Prove that all rays which after reflexion pass
gh the line $x=y=z$, were before reflexion in the surface
cone defined by the equation

$$yz\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+zx\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+xy\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0.$$

. A pencil of rays is incident parallel to the axis x of a
ting ellipsoid whose greatest and least axes are a and c
tively; find the nature and *limits* of the two curvilinear
aries of the portion of the plane x, y through which all
ys will pass;—and shew that if e, e', e'' be the eccen-
s of the principal sections in the planes xy, xz, yz re-
vely, and $\mu=\frac{1}{e}$, the boundaries will be two ellipses
semi-axes are ae''^2, be''^2, and $\frac{cae}{b}$, be''; but if $\mu=\frac{1}{e'}$, all
ys will pass through a *portion* of the *arc* of an ellipse,

who 1-axes are ae', be'' included between the vertex and a do nate at a distance $\frac{cae'}{b}$ from its centre.

22. surface of a piece of water is covered, except one narrow slit in the form of a straight line: a luminous point is placed in a given position above the surface;—if this be taken as origin of co-ordinates and the vertical line as axis of z; and $x=a$, $z=-c$ are the equations to the slit,—shew that the equation to the sheet of light in the water is

$$a^2\{(x^2+y^2)(x-a)^2+x^2(z+ \quad\quad (x-a)^2\{a^2(x^2+y^2)+c^2x^2\}.$$

23. Rays diverging oint a, b, c are reflected at the curve $z=0$, $f(x, y)=$ rmals to the polished part of which all lie in the plan ; shew that the equations to the ray reflected at the point y, 0) are

$$\frac{X-x}{2p(b-y)-(p^2-1)(a-x)}=\frac{Y-y}{2p(a- \quad)+(p^2-1)(b-y)}=\frac{Z}{(p^2+1)c}$$

where $p=\frac{dy}{dx}$,—and the sheet or surface formed by the reflected rays will be found by eliminating x, y between these equations and $f(xy)=0$.

Ex. If $f(x, y)=y^2-4ax$ and $b=0$, the ray-surface of reflexion is

$$Y^2(c-Z)=4a(aZ+cX).$$

* * * * *

The following examples of *ray-surfaces of reflexion* formed as in problem (23) are taken with modifications from an interesting work on *Ray Surfaces of Reflexion*, by Prof. Childe, Cape Town, 1857. I am indebted to the same work for the problems 31—34 of this Chapter. *Lateral* and *vertical* ray-surfaces of reflexion are the names given to surfaces formed as in problems 23 and 31 respectively.

24. In problem (23)

(i) The equation to the plane passing through the incident ray and the normal to the reflecting curve is

$$X-x+(Y-y)p-\{a-x+(b-y)p\}\frac{Z}{c}=0.$$

(ii) The equation to the plane passing through the reflected ray and the tangent to the reflecting curve is

$$(X-x)\,p-(Y-y)+\{(a-x)\,p-(b-y)\}\frac{Z}{c}=0.$$

25. In (23) if the incident rays are all parallel, their direction being defined by the direction-cosines l, m, n, the equations of the reflected ray become

$$\frac{X-x}{2mp-l\,(p^2-1)}=\frac{Y-y}{2lp+m\,(p^2-1)}=\frac{Z}{(p^2+1)\,n}.$$

In this example, if the ratio $l:m$ remain constant whilst n varies, the resulting ray-surfaces of reflexion will have for their envelope the cylinder whose axis is parallel to the axis of z, and whose trace on the plane of xy is the ordinary caustic formed by the reflexion of rays all incident in the direction defined by $(l, m, 0)$.

26. If $f(xy)=y-x\tan\alpha=0$, the ray-surface of reflexion (either *lateral* or *vertical*) is

$$\frac{Z}{c}+\frac{X\sin\alpha-Y\cos\alpha}{a\sin\alpha-b\cos\alpha}=0.$$

27. If $f(xy)=x^2+y^2-\rho^2=0$ a circle,—the ray-surface of reflexion is

$$\{c^2(x^2+y^2)-(a^2+b^2)\,z^2\}^2=\rho^2\,(z+c)^2\,\{(cx-az)^2+(cy-bz)^2\}.$$

28. In the preceding example if the incident rays are all parallel, and their directions defined by (l, m, n) the ray-surface of reflexion is

(i) $\{n^2(x^2+y^2)-(l^2+m^2)\,z^2\}^2=\rho^2\,\{(nx-lz)^2+(ny-mz)^2\}$

or writing $m=0$—which does not sacrifice any generality of form—this becomes

$$\{n^2(x^2+y^2)-l^2z^2\}^2=\rho^2\,\{(nx-lz)^2+n^2y^2\}\,\ldots\ldots\ldots$$

(ii) If the direction of this system of parallel rays be varied, subject to the conditions $m=0$, $l^2+n^2=1$, only—the envelope of the ray-surfaces given by (i) is

$$(\rho^2-x^2-y^2)\,\{8\,(x^2+y^2)+\rho^2\}^2=27\rho^4x^2.$$

This en[illegible] is an epicycloidal cylinder whose axis is parallel to the axis of z, and [illegible] trace on xy is the epicycloid generated by a circle of radius $\frac{r}{4}$ rolling on a circle of radius $\frac{r}{2}$, the cusps of the epicycloid being on the axis of x. It is in fact the epicycloid of Art. 71.

29. Find the envelope of the *lateral* ray-surfaces of reflexion, the reflecting curve being (as in 27)—the circle

$$x^2+y^2-r^2=0,$$

and the locus of the radiant point (a, b, c) a circular ring defined by $c=$ constant, $a^2+b^2=d^2$.

Result. The double [illegible] by the equation

$$c\sqrt{(x^2+ [illegible] \pm r(z+c).$$

30. If the reflecting curv[illegible]s in 28) be a cycloid

$$z=0, \qquad y= [illegible] \sqrt{(2ax-x^2)};$$

and the incident rays are all par[illegible]el to the plane of xz, their directions being defined by $(l, 0, n)$—the *lateral* ray-surface of reflexion is

$$ny=na \operatorname{vers}^{-1}\left(\frac{lz+nx}{lz+na}\right)+\sqrt{[(lz+nx)\,[lz-n\,(x-2a)]]}.$$

This surface is discussed at length in the work above referred to.

* * * * *

31. Rays diverging from a point (a, b, c) are reflected at a polished curve whose equations are $z=0$, $f(x, y)=0$, the normals to which are all parallel to the axis of z: shew that the equation to the surface formed by the reflected rays—*the ray-surface of reflexion*—is

$$f\left(\frac{cX+aZ}{c+Z},\ \frac{cY+bZ}{c+Z}\right)=0.$$

32. In the preceding question, if the incident rays are all parallel—their direction being defined by the direction-cosines l, m, n—the ray-surface of reflexion will be

$$f\left(\frac{nX+lZ}{n},\ \frac{nY+mZ}{n}\right)=0.$$

If the polished curve in problem (31) be a circle, *viz.*

$$f(xy) = 0 = x^2 + y^2 - \rho^2,$$

y-surface of reflexion is

$$(cx + az)^2 + (cy + bz)^2 = \rho^2 (z + c)^2.$$

:. Find the envelope of the ray-surfaces of reflexion 1 by the curve $z = 0$, $x^2 + y^2 - \rho^2 = 0$, as in problems 31 3, when the locus of the radiant point a, b, c is

A circular ring defined by the equations $c =$ constant, $= k^2 =$ a constant.

sult. The double cone defined by the equation

$$c\sqrt{(x^2 + y^2)} + xz = \pm \rho (z + c).$$

) A circular ring defined by $c =$ const., and

$$(a - \alpha)^2 + (b - \beta)^2 = \kappa^2, \quad \alpha, \beta, \kappa \text{ constants.}$$

sult. Two oblique cones defined by

$$(cx + \alpha z)^2 + (cy + \beta z)^2 = \{(\rho \pm \kappa) z + \rho c\}^2.$$

i) A luminous spherical surface concentric with the ing circle and defined by the equation $a^2 + b^2 + c^2 = \kappa^2$.

sult. Two right cones defined by the equation

$$\{\sqrt{(x^2 + y^2)} \pm \rho\}^2 = \frac{\rho^2 - \kappa^2}{\kappa} . z^2.$$

) A luminous paraboloid of revolution, *viz.*

$$a^2 + b^2 = 4mc.$$

sult. The envelope is a cone,

$$\rho^2 (x^2 + y^2) = (mz - \rho^2)^2.$$

the *fixed line A* in its axis, and the *fixed line B*, having given $\mu_A = \sqrt{3}$, $\mu_B - \mu_A = \cdot 001$, the refracting angle of the prism being $= 60^\circ$ and the distance of its edge from the slit and the object-glass being 3 and 10 times the focal length of the object-glass respectively,—the telescope magnifying 20 times.

With the notation of Art. 168, the angle which the lines A and B subtend at the prism $= \frac{\sin i}{\cos\phi' \cos\psi}\,\delta\mu = \cdot 002$ in circular measure, after reduction.

If f = focal length of the object-glass, the linear distance of A and B in the image formed by the prism $= 3f.(\cdot 002) = \cdot 006.f$. This distance subtends at the object-glass, the $\angle \frac{\cdot 006}{13}$, and therefore the corresponding part of its image formed by the telescope, and viewed by the eye subtends at the eye the $\angle \frac{\cdot 006}{13}.20 = \cdot 00923$ in circular measure $= 31'44''$ nearly.

23. If a small pencil of light pass directly through a plate of thickness b, the index of refraction being $f\left(\frac{x}{c}\right)$, x being measured from the plane of incidence, and c varying slightly with the colour of the light,—shew that the chromatic aberration on emergence is

$$\left\{\frac{b}{c}\cdot\frac{1}{f\left(\frac{b}{c}\right)} - \int_0^b \frac{dx}{cf\left(\frac{x}{c}\right)}\right\} dc;$$

$f\left(\frac{0}{c}\right)$ being supposed equal to unity.

CHAPTER X.

Vision through Lenses, &c.

1. A person can see distinctly at the distance of 6 inches, find the focal length and nature of a lens which will enable him to see distinctly at a distance of 18 inches.

2. A concave lens is placed directly between an eye and a screen, determine how much of the screen will be visible *through* the lens.

7. A direct pencil of rays converges to a point at the extreme distance from a lens at which an absence of aberration in a converging pencil is possible: determine the relation between the radii of the lens that there may be no aberration in the refracted pencil. $\left(\mu=\frac{3}{2}\right)$.

Result. With the notation of Art. 132, we must have

$$\frac{\mu+2}{\mu-1}x^2-4(\mu+1)ax+(\mu-1)(3\mu+2)a^2+\frac{\mu^3}{\mu-1}=0,$$

which gives

$$x=\frac{2(\mu^2-1)a\pm\mu\sqrt{(\mu-1)^2a^2-\mu(\mu+2)}}{\mu+2};$$

the limiting value of a is $\quad a=\pm\frac{\sqrt{\mu(\mu+2)}}{\mu-1}=\pm\sqrt{21}=\pm 4{\cdot}58$;

the pencil being convergent at incidence, u is negative, and therefore the negative value of a must be taken, and we obtain

$$x=2\frac{\mu^2-1}{\mu+2}a=-3{\cdot}27,$$

$$\text{and } \frac{r}{s}=\frac{x-1}{x+1}=\frac{427}{227}=\frac{32}{17} \text{ nearly.}$$

8. A pencil of parallel rays falls directly upon a double equiconvex lens: shew that if the lens be split in two by a plane perpendicular to its axis and the two parts be separated by an interval equal to $\frac{2}{3}$ the focal length of either, the convergency of the emergent pencil is double that in the former case when the lens is entire.

9. Three convex lenses are placed on the same axis; shew that the focal length (F) of the equivalent single lens is given by the equation,

$$f_1f_2+f_2f_3+f_3f_1-a(f_1+f_3)-b(f_1+f_2)+ab=\frac{f_1f_2f_3}{F},$$

where $f_1f_2f_3$ are the focal lengths of the three lenses, and a, b the distances between the first and second, and the second and third respectively.

one image is erect and the other inverted,—shew that their mean distance from the lens is equal to its focal length.

11. A small object is viewed through two convex lenses of focal lengths f_1, f_2 separated by an interval $=m(f_1+f_2)$, m being a proper fraction;—find the position of the object when most distinctly seen, and shew that the magnifying power of the combination is

$$a(1-m)\left(\frac{1}{f_1}+\frac{1}{f_2}\right),$$

where a is the least distance of distinct vision with the naked eye.

12. Shew that a defect in the eye of such a nature that the rays of a pencil incident in a horizontal plane are differently refracted from those in a vertical plane, may be remedied by the use of a lens of which one surface is cylindrical and the other spherical. Find the foci of a pencil of parallel rays in the two principal planes, when the radii of the sphere and cylinder are $3\frac{1}{2}$ and $4\frac{1}{2}$ inches respectively.

13. A closed hollow cylinder, about two inches long, has in the middle of one end a very small hole, and in that of the other a circular aperture of about the same diameter as the pupil of the eye. A pin is so placed in the plane of the aperture that its head is near the centre. When the flame of a candle is viewed through the tube, the aperture being held close to the eye, the flame appears upright and the pin inverted.

Explain the phenomenon.

Telescopes, &c.

14. The angular radius of the uniformly bright field of view in Galileo's Telescope—with the usual notation—is

$$=\frac{f_e y_0 - f_0 y_e}{f_0(f_0-f_e)}.$$

15. The aperture of the object-glass of an Astronomical *Telescope* being 5 inches, and that of the pupil of the eye $\frac{1}{5}$ of

will be parallel, (ii) the position of the luminous point along the axis which renders the distance of these points from the axis a *maximum* or *minimum*.

5. A circular luminous ring, radius a, is placed on the ceiling of a room;—in the floor is a hole in the form of an equilateral triangle of side b; shew that the area of illuminated patch on the floor of the room beneath

$$=\pi a^2+\sqrt{3}\,b^2+6ab,$$

the rooms being of equal height.

6. Psp is a focal chord of a parabola, S a luminous point; shew that the illumination at P : illumination at p :: $(Sp)^{\frac{5}{2}} : (SP)^{\frac{5}{2}}$.

7. Two candles (P, Q) of equal intensity situated at the same distance from a wall, throw upon it shadows (m, n) of a small object; prove that the illumination in these two shadows are as $Pn^3 : Qm^3$.

8. Two candles are placed on a horizontal table, the one lighted, the other not,—the latter being the taller and fixed: the other is moved about on the table so that the shadow of the first on the ceiling is of constant length. Shew that the locus of the foot of the candle is a conchoid.

9. The circumference of a circle is luminous except a very small arc which serves as a reflector; find the image of the luminous part when the image is defined to be the locus (i) of the primary focal lines, (ii) of the secondary focal lines, (iii) of the circles of least confusion.

10. The curvature at the vertex of the image of a straight line formed by refraction through a sphere $=2\,\dfrac{\mu-1}{\mu}$. curvature of the sphere.

11. A parabola whose latus rectum $=2l$ is placed before a concave spherical refracting surface, the focus and axis of the parabola being coincident with the centre and axis of the surface. The parabola is concave to the surface, and an

image of it is formed by direct pencils. Shew that if f be the distance of the principal focus from the surface (f being $> l$) the image will be a confocal hyperbola whose eccentricity is $= \frac{f}{f-l}$.

12. In the flat roof of a building there is a square hole exposed to the light of a uniform sky. The floor of the uppermost room has also a square hole in it under the former but its angles opposite to the sides of the former:—determine the form of the whole illuminated surface of the floor of the room below, and the illumination at any point.

13. A straight line passes through the point of intersection of three rectangular reflecting planes, and is equally inclined to each of them; find the series of images produced, and distinguish between those resulting from one, two, and three reflexions respectively.

14. The image of a straight line perpendicular to the axis of a convex lens at a very great distance from it, approximates to a parabolic curve, whose equation is

$$2(x+f) + \left(2 + \frac{1}{\mu}\right)\frac{y^2}{f} = 0,$$

the centre of the lens being the origin, and the circle of least confusion being taken to be the image of any point.

15. Shew that the image of the surface $f(x, y, z) = 0$ made by reflexion at the plane mirror $lx + my + nz = 0$ is $f(x + lp, y + mp, z + np) = 0$, p being such a quantity that $lx + my + nz + \frac{p}{2} = 0$.

16. An image of a very distant object is formed by a plano-convex lens, the pencils before incidence on the lens passing through a small diaphragm (B), whose centre is in the axis (ABO) of the lens: O being the centre of the convex surface, and A the point where the axis meets the plane surface of the lens. If $AO = \mu \,.\, AB$, the image formed by *the lens* will be *distinct*.

N.B. *Distinctness* will be secured if the refraction at the second surface—the *convex* one—be *direct*.

17. A stop is placed on the axis of a concave spherical reflector at a distance from it equal to four times its focal length: to measure the distortion at any point of the image of a very distant object to which the axis is directed.

With a figure and notation similar to that of Art. 150 (the positions of B and Y must be reversed)—if Y be the stop or diaphragm, $AR=y$, q being the point of the image seen corresponding to the point Q of the object, we shall obtain

$$qm=\frac{r}{2}\tan\epsilon\left(1+\frac{3y^2}{8r^2}\right),$$

now the part of the object whereof qm is the image has a size proportional to $\tan\epsilon$; hence the distortion at any point of the image is measured by the term involving y^2 in the expression

$$\frac{r}{2}\left(1+\frac{3}{8}\frac{y^2}{r^2}\right).$$

18. A small light is placed at the focus of a perfect reflector in the form of a paraboloid of revolution: prove that the brightness, due to reflection, at any point within the volume of the paraboloid, varies inversely as the square of the focal distance of the end of the diameter through the point.

19. Light is transmitted from a very distant luminous sphere through an orifice in an opaque screen A, and is received on a second screen B parallel to A:—shew that

(i) if the distance of B from A be small, compared with the size of the orifice, the illuminated figure on B will resemble the orifice in A.

(ii) if the distance of B from A be large, the illuminated figure on B will be nearly circular.

Caustics.

20. The perimeter of an ellipse is luminous except a very small arc at the extremity of the major axis, determine the caustic.

1 If light be incident from a point Q on any surface, a c be he half chord of curvature at any point of the su direction of the incident ray, then $\frac{x}{2}$ is a mean proportional between $u-\frac{x}{2}$ and $v-\frac{x}{2}$.

If the uation to the surface be given in the form $u=\phi(x)$, en the arc of the caustic can be found from the equation

$$(2u-x)^2 \qquad x)\,du+2u^2dx.$$

22. Parallel rays it parallel to an asymptote on a reflecting rectang bola: find the equation to the caustic curve.

If ρ be the ra re of the caustic, ρ_1 of the corresponding poin bola, shew that the brightness of the caustic $\propto \frac{\rho}{(\rho_1)^{\frac{2}{3}}}$.

23. Rays emanate from th pole of a plane curve whose equation is given in the form $f(r,p)=0$ (i); shew that the equation to the catacaustic will be the result of eliminating r and p between (i) and the equations

$$\sqrt{(r'^2-p'^2)}+\sqrt{(r^2-p'^2)}=\frac{rp\frac{dr}{dp}}{2r-p\frac{dr}{dp}}\ldots\ldots\ldots\ldots\text{(ii)}.$$

$$p'^2=\frac{4p^2(r^2-p^2)}{r^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Conversely, if $\phi(r',p')=0$, (iv), be the equation to the catacaustic, the equation of the reflecting curve will be obtained by eliminating $r'p'$ between (ii), (iii), and (iv).

Examples.

(i) If $r^2=p^2+a^2$,—the involute of a circle,—be the reflecting curve, the caustic is $\frac{r}{2a}=\frac{\sqrt{\{16a^4+p^2(4a^2-p^2)\}}}{8a^2-p^2}$.

(ii) If $r\theta = a$, or $(r^2 + a^2)p^2 = a^2r^2$,—the hyperbolic spiral, -be the reflecting curve, the caustic is $r = \frac{ap}{a-p}$.

(iii) If $r = a$—a circle—be the equation to the caustic, the quation to the reflecting curve is $ar = 2p\sqrt{(r^2 - p^2)}$.

24. If rays emanating from the vertex are reflected from parabola, the caustic is the evolute of a cissoid.

25. Rays emanating from the focus of a parabola are eflected from the evolute of the parabola—the caustic is the volute of a parabola.

26. The caustic of a cardioid $r = a(1 + \cos\theta)$,—the pole eing the radiant point,—is an epicycloid due to a circle of adius $\frac{a}{4}$ rolling on a circle of radius $\frac{a}{2}$.

27. A ray whose direction is parallel to the axis of x is ncident on a point xy of a refracting curve; prove that the orresponding point of the caustic $(x'y')$ is given by the equa-ions

$$x' = x - \frac{\mu^2(1+p^2) - 1}{\mu^2 q}\left[\frac{1 + p\sqrt{\{\mu^2(1+p^2) - 1\}}}{\sqrt{\{\mu^2(1+p^2) - 1\}} - p}\right],$$

$$y' = y + \frac{\mu^2(1+p^2) - 1}{\mu^2 q},$$

where $p = \frac{dy}{dx}$, $q = \frac{d^2y}{dx^2}$.

A ray whose direction is parallel to the axis of x is inci-ent on a parabola $x^2 = 2ay$ at the point $x = y = 2a$, and is fracted into a medium for which $\mu = \sqrt{2}$,—find the corre-onding point on the caustic.

CHAPTER IX.

Of the Chromatic Dispersion of Light.

1. In Newton's experiment—supposing a second prism to be placed behind the first with its edge perpendicular to it, what will be the effect upon the spectrum?

2. When objects are viewed by means of pencils of light which are refracted through an isosceles prism, suffering an internal reflexion at the base, the objects are seen quite free from chromatic dispersion, provided the prism be not very large; prove that this must happen in the general case,—that is, when the refraction is not supposed to take place in a plane perpendicular to the edges of the prism.

3. If a prism be laid on its base in the open air and the eye be placed in a proper position, the base will appear to be divided into two parts, the one much brighter than the other, and separated from one another by a coloured bow (*blue and violet*) concave towards the eye. Give *Newton's* explanation of the cause of this phenomenon.

Newton, *Optics*, Part II. Prop. 8.

4. An opaque rod held parallel to the edge of a glass prism is viewed by an eye applied to the prism,—the rod appears curved with its concavity towards the edge,—and fringed with red and yellow on its concave side, and with violet and blue on the convex side. Explain these phenomena.

5. Two plates of different dispersive powers, but in both of which the index of the extreme violet rays is the same, are placed in contact, and light is obliquely incident upon them. Prove that the reflected rays will be tinged with red, and the transmitted rays with violet.

6. A black line is painted on the base of an isosceles *prism*, midway between its two edges, and the prism is then

interposed between the sun-light and an eye held at the edge of the prism looking towards the base. Describe the appearances presented with respect to position and colour.

7. A luminous point of white light being placed on the axis of a lens,—shew that as the point moves along the axis the distance between the geometrical foci for two given colours will pass through a minimum,—and determine the position of the point in this case.

8. A star, the altitude of which is α, is viewed by an eye under water; shew that it will appear in the form of a spectrum, with the violet end uppermost, and that the angular magnitude of the spectrum will be

$$\sin^{-1}\left(\frac{\cos\alpha}{\mu_r}\right)-\sin^{-1}\left(\frac{\cos\alpha}{\mu_v}\right).$$

9. An achromatic object-glass, whose focal length is 70 inches, is composed of two lenses of crown and flint-glass respectively; find the focal length of each lens, the dispersive powers of crown and flint-glass being in the ratio of 7 : 10.

10. For a prism A of small refracting angle the indices for two colours are 1·2 and 1·22, and for a prism B are 1·5 and 1·57; find the angles of the prisms in order that the common deviation of the two colours—passing through the prisms in a principal plane of each—may be $=1°$.

11. A system of three convex lenses of the same substance on the same axis will be achromatic for rays incident parallel to the axis, if

$$3ab-2a(f_2+f_3)-2b(f_1+f_2)+f_1f_2+f_2f_3+f_3f_1=0;$$

a, b being the distances of the middle lens—focal length f_2—from the two extreme ones whose focal lengths are f_1, f_3.

12. A ray of light after passing through a prism at the angle of least deviation falls perpendicularly on the surface of

another prism of different substance, shew that the ray will be achromatised if the angle between the prisms

$$= \cos^{-1}\left(2m \sin \frac{i}{2} . \cot i_1\right),$$

where m is the ratio of the indices of refraction and i, i_1 the refracting angles of the prisms—the ray passing in a principal plane of each.

13. A small pencil of rays of common light is incident obliquely on a plane refracting surface, shew that the primary foci of the refracted pencils of different colours will lie on a curve of the third order.

14. Find the condition of achromatism of an eye-piece composed of a solid block of glass with a thin lens of different glass cemented to it on the end next the eye,—the surfaces being worked spherical.

15. Find the condition of achromatism of an eye-piece composed of a single block of glass worked at the ends into spherical surfaces and used with an object-glass of great focal length.

16. A compound object-glass is to be formed of two lenses in contact; shew that if when the lenses are ground, achromatism is nearly but not quite secured, the defect may be remedied by slightly separating the lenses.

The refractive indices corresponding to the letters D and F in the *orange* and *blue*, for certain kinds of crown and flint glass are

Crown-glass . . .	1·5279,	1·5344,
Flint-glass . . .	1·6351,	1·6481,

twenty inches is to be the focal length of the proposed object-glass; find the focal lengths of the two lenses which, placed in contact, unite these *lines*.

17. Trace the variation in the angular magnitude of the spectrum formed by a prism, for different angles of incidence, and shew that it is a minimum when, with the usual notation,

$$\mu^2 \sin(\phi' - i) \cos(2\phi' - i) + \sin\phi' = 0.$$

7. A person having a very small fragment of a concave reflector, the principal radii of curvature of it being ρ, ρ', wishes to place it so that it may be considered part of a paraboloid of revolution, the positions of the focus and axis of which are given. Shew that it must be placed with its plane of least curvature passing through the focus, at the distance $\frac{1}{2}\sqrt{(\rho\rho')}$ therefrom, and with its tangent plane inclined to the axis of the paraboloid, and to its focal distance at an angle whose sine is $\sqrt{\left(\frac{\rho'}{\rho}\right)}$.

8. It has been remarked by writers on optics that "the concavity of the heavens appears to the eye to be a less portion of a spherical surface than a hemisphere,"—and that "the apparent distance of its parts at the horizon is generally between three and four times greater than the apparent distance of its parts overhead." How do you account for this appearance of the sky? By what method are the proportions here mentioned determined? Shew how these circumstances may be used in accounting for the apparent change in the size of the sun or moon, or in the distance of two neighbouring stars, as they ascend from the horizon to the meridian.

9. When a cylindrical china jar, standing upon the ground, receives the sun's rays obliquely, a bright curve is observed to form itself at the bottom of the jar, and it is found that the shape and dimensions of this curve are not affected by the varying elevation of the sun: account for this latter circumstance, and determine the nature of the bright curve.

10. A plane mirror, moveable about an axis in its own plane parallel to the axis of the earth, revolves from east to west with half the sun's apparent diurnal motion. Shew that the direction of the reflected rays of sunlight will not be sensibly altered during the day.

11. The sun's light is refracted through a prism, the edge of which is vertical,—find the position of the refracting surfaces in order that for a given altitude of the sun the deviation of the rays of a given colour may be a minimum.—

If z be the sun's zenith distance, i the refracting angle, x the angle of first incidence reduced to the horizon, μ the index for the given colour,—shew that the minimum deviation D is given by the equations

$$\sin\frac{D}{2}=\sin z\sin\left(x-\frac{i}{2}\right),$$

$$\sin x=\mu\sin\frac{i}{2}\cdot\sqrt{\left\{1+\left(1-\frac{1}{\mu^2}\right)\cot^2 z\right\}}.$$

12. One half of a circle is a bright reflecting surface, the other half dull; a luminous point is placed at the bisection of the dull part. Shew that the bright surface is divided into two portions each of which is equally illuminated.

13. A narrow flat polished steel-wire is bent into the form of a circle, so as to form a cylinder of indefinitely small height; a luminous point is placed so that the perpendicular from it to the plane of the ring falls within the ring. Shew that all the rays after reflexion pass through a straight line of finite length. Calculate the length and position of this line, and point out the modification of the problem when the perpendicular does not fall within the ring.

14. If a plane surface be placed parallel to the plane of the ring—(Prob. 13)—and below it, the bright curve on the plane surface has for its equation $r=a+b\cos\theta$, the point where the line passing through the luminous point and the centre of the ring meets the plane being taken as the pole.

Trace this curve and find the condition that it may have a cusp.

15. P, Q are two points on an ellipse; from the pole of PQ a normal is drawn to the curve cutting it in R; shew that a ray from P incident at R will be reflected to Q.

16. A *small* plane area is illuminated by a bright circular ring of given dimensions,—the line joining the centres of the area and of the ring being perpendicular to the plane of the ring; find the position of the area, so that the illumination may be the greatest possible.—

If the area be replaced by a bright point, find its position so that the greatest quantity of light may be thrown on the ring.

17. A sphere of glass encloses a concentric sphere of an opaque substance and is placed on a horizontal table, a bright point is placed very near, but not close to its highest point; find the least magnitude of the opaque sphere which will prevent any ray from reaching the table;—if the magnitude be slightly less than this, find the diameter of the circular bright ring on the table.

18. A ray of light is reflected a number of times between two plane mirrors—not in a principal plane; prove that all the reflected segments of the ray are generating lines of a hyperboloid of revolution.

19. A plane mirror revolving about a vertical axis in its own plane, receives and reflects a small sunbeam, which after reflexion forms a spot of light on the horizontal floor. Determine the motion of the spot on the floor, and shew that its image as seen in the revolving mirror is stationary.

20. A small plane mirror is placed at the principal focus of a telescope, nearly perpendicular to its axis, and the telescope is directed approximately to a distant luminous object; shew that the rays reflected at the mirror will, after repassing the object-glass, return in the exact direction in which they came, in spite of the small errors of adjustment of the mirror and telescope.

21. A surface exposed to a luminous sphere is such that the illumination at any point $\propto$ (distance)$^{-n}$ of the point from the centre of the sphere. Examine the cases when $n = 0, 1, 2, 3$.

22. The outer surface of a transparent shell is an ellipsoid, and the inner surface is a concentric sphere. A luminous point is placed at the centre; find the greatest value of the index of refraction of the medium which will allow every ray to emerge without suffering internal reflexion.

from the uniformly bright surface of the cube; shew that the illumination is measured by the expression

$$3I.\frac{\theta}{\sin\theta}\sin\alpha\sin\beta\sin\frac{\pi}{3},$$

where θ is the angle between two adjacent edges of the pyramid (vertex O) circumscribing the cube, and α, β are their inclinations to the diagonal.

27. Two parallel and equal lines AB, CD are traced on paper, and viewed by the two eyes placed so that the line joining the centres of the eyes is parallel to AC; shew that the eyes may be so adjusted that the images of AB, CD may appear superposed, and that the resulting image will appear parallel to the plane of the paper, nearer to the eyes and smaller than AB or CD.

Under the same circumstances if AB, CD be inclined at a small angle, their images may appear superposed, and the resulting image will appear to *project* from the plane of the paper. Explain this, and shew why the parts of two pictures which, when viewed with the *stereoscope*, appear to project towards the eyes, frequently appear, when superposed by the naked eyes, to project in the opposite direction.

28. The shadow of a ball is cast by a candle upon an inclined plane in contact with the ball; prove that as the candle burns down, the locus of the centre of the shadow will be a straight line.

29. The refractive index at every point of a certain transparent medium is a *certain* function of the co-ordinates of that point; shew that if a particle move in such a manner that its velocity at any point is proportional to the *same* function of the coordinates, the particle may be so projected, that its path shall be the same as the course of a given ray through the medium.

30. On what law of density will the total refraction of a ray of light through the Earth's atmosphere be the same as through a homogeneous atmosphere?

31. A vertical window-bar of an opposite house, seen through a pane of glass, appears a zig-zag line; describe the inequalities in the thickness of the pane of glass indicated by the deviations from the vertical, and give reasons for your conclusion.

Rain is streaming down a pane of glass in parallel vertical lines; what will be the appearance of a circular arch seen through it?

32. The shadow of a given ellipsoid thrown by a luminous point on the plane which passes through two of the principal axes, has its centre on the curve in which the same plane intersects the ellipsoid; shew that the equation to the locus of the luminous point is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{z^2}{c^2}-1\right)^2,$$

where a, b the semiaxes of the ellipse through which the plane passes, and c the remaining axis, are taken for the axes of x, y, z respectively.

33. A polished hemisphere is placed with its base in contact with a plane, and a cylindrical pencil of rays falls upon its convex surface in a direction perpendicular to the plane: the illumination at any point of the plane is proportional to

$$\frac{\sin^3\phi}{2+\sin\phi},$$

where ϕ is the angle which the ray reflected to that point makes with the plane.

34. A uniformly bright cylindrical column, of radius a, and of indefinitely great height, stands upon a plane; shew that the illumination of a disc of the plane, of radius r, concentric with the column,

$$\propto \frac{\pi}{2}(r^2-a^2)+a\sqrt{(r^2-a^2)}-r^2\cos^{-1}\frac{a}{r}.$$

35. Denoting two mirrors by $\alpha=0$, $\beta=0$, what must be

the angle between them, that the ray parallel to $\alpha-\beta=0$ may, after reflexion at each mirror, be parallel to $\alpha+\beta=0$.

36. The illumination at any point of an equilateral hyperbola, the centre being the origin of light, $\propto$ (distance)$^{-4}$.

37. Shew that the space-penetrating power in an astronomical telescope is measured by $\left(A\,.\,\dfrac{f}{F}\right)^2$, where F and f are the focal lengths of the object and eye-glass respectively.

38. The magnifying power of a telescope is

$$=\frac{\text{breadth of visual pencil at the object-glass}}{\text{breadth of visual pencil at the eye-glass}},$$

supposing no light lost at any lens.

39. If S be a point from which rays fall on a plane curve, H any point in the ray reflected from the element of the curve at P; if HE be drawn parallel to the tangent at P cutting SP in E, then will

$$SP+PH=2\,.\,PE.$$

40. An erecting eye-piece of four lenses has the focal lengths of the lenses f_1, f_2, f_3, f_4 and their intervals a_1, a_2, a_3; if f is the focal length of the equivalent simple lens,

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}+\frac{a_1}{f_1}\left(\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}\right)+a_2\left(\frac{1}{f_1}+\frac{1}{f_2}\right)\left(\frac{1}{f_3}+\frac{1}{f_4}\right)$$
$$+\frac{a_3}{f_4}\left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}\right)+\frac{a_1a_2}{f_1f_2}\left(\frac{1}{f_3}+\frac{1}{f_4}\right)+\frac{a_1a_3}{f_1f_4}\left(\frac{1}{f_2}+\frac{1}{f_3}\right)$$
$$+\frac{a_2a_3}{f_3f_4}\left(\frac{1}{f_1}+\frac{1}{f_2}\right)+\frac{a_1a_2a_3}{f_1f_2f_3f_4}.$$

41. A luminous sphere rests within a hemisphere of twice its radius, the rim of which is horizontal; find the whole illumination of the interior surface of the hemisphere;—and if the sphere be raised so that its lowest point just coincides

with the centre of the hemisphere, show that the [illegible] will be [illegible] in the ratio

$$\sqrt{5} - 1 : \sqrt{5} - 1.$$

42. A pencil of rays is refracted directly through a series of concentric spherical surfaces: if μ_{n-1}, μ_n be the indices of refraction from vacuum into air and into the n^{th} medium, counting from the outermost, r_n the outer radius of the n^{th} surface, and v_0, v_n the distances of the initial focus and of the focus after n refractions from the common centre, show that

$$\frac{1}{\mu_0 v_0} - \frac{1}{\mu_n v_n} = \Sigma \frac{\mu_{n-1} - \mu_n}{\mu_{n-1}\mu_n r_n}.$$

43. A ray of light is incident perpendicularly on one of the faces of a prism of which the density varies in such a manner that the coefficient of refraction at any point is μe^{θ}, μ being constant, and θ the angle which a plane through the point and the edge of the prism makes with that face upon which the ray is incident. If α be the refracting angle of the prism, ϕ the angle of incidence on the second face, show that ϕ is determined by the equation

$$\cos\phi - \sin\phi = e^{[illegible]}.$$

44. The focal lengths of the large and small mirrors of a Gregorian telescope are 18 and $\frac{19}{20}$ inches respectively, and their distance from each other is 19 inches: find the position of the image formed by the small mirror, and the magnifying power, if the focal length of the eye-glass $= \frac{1}{4}$ inch.

45. The density at any point of a prism (vertical $\angle = 2\beta$) varies as its distance from the nearest face of the prism. If a ray pass through it in a principal plane, its distance from the edge at the points of incidence and emergence being a, and its nearest approach to the edge being c, the deviation D is given by the equation

$$\sin\left(\beta + \frac{D}{2}\right) = \sin\beta \, . \, e^{\frac{Dc\sin\beta}{2(a - c\cos\beta)}}.$$

46. A spherical mirror is to be reduced by grinding to

a parabolic one of the same focal length; find approximately what thickness parallel to the axis must be ground away at any point.

47. In Art. 122, if $\mu=f(x)$ shew that

(i) the curvature at any point of the trajectory

$$\propto \frac{d}{dx}\cdot\left\{\frac{1}{f(x)}\right\}.$$

(ii) find the form of $f(x)$ in order that the trajectory may be a hyperbola.

(iii) if $f(x)=\alpha+\beta\sin\frac{x}{c}$, determine the trajectory.

48. A small pencil is *directly* refracted through a medium or which the index of refraction at any point is $\mu=\frac{ak}{d+x}$ (as in Art. 122). Shew that the distance of the geometrical oci from the origin of light will be

$$=\tau\left(1-\frac{2d+\tau}{2ka}\right),$$

here τ is the whole thickness of the medium traversed.

49. The distance between a luminous point and the entre of the pupil of the eye is (D), and the radius of the upil (r), which is supposed very small compared with D; hew that the illumination of the pupil is

$$\frac{c\pi r^2}{D^2}\cos\alpha\left(1-\frac{3}{4}\frac{r^2}{D^2}+\frac{15}{8}\frac{r^2}{D^2}\sin^2\alpha\right)\text{ nearly},$$

here α is the angle which D makes with the axis of the eye, nd higher powers of $\frac{r}{D}$ are neglected.

50. The axis of the object-glass of an astronomical telescope is inclined to the axis of the telescope at a small angle α;

55. If N, n, F be the focal centres and principal focus of a lens, the distances of the conjugate foci measured from two points N', n',—so situated in the axis that $nn' = p \,.\, NN' = (1-p) \,.\, nF$,—are connected by the equation

$$\frac{p}{v} - \frac{1}{pu} = \frac{1}{nF};$$

p being any constant quantity.

56. A hollow paraboloidal vessel, whose upper rim is circular, is placed with its axis vertical upon a horizontal plane, and exposed to the Sun's rays. The boundary of the illuminated part of the interior of the vessel is a plane curve, whose projection on the plane of the rim is a circle of the same radius as the rim, and whose centre is distant from the centre of the rim a space equal to the latus rectum of the vessel multiplied by the tangent of the Sun's altitude.

57. The distance between two lenses of a common astronomical telescope in perfect adjustment is a, the magnifying power being $\frac{q}{p}$. It is found that the object-glass can be achromatised by means of a certain lens placed in contact with it, and that if at the same time a concave lens of focal length (b) be placed in contact with the eye-glass the instrument will still be in adjustment; shew that if Δ_1, Δ_2 be the dispersive powers of the media forming the compound object-glass,

$$\frac{\Delta_1}{\Delta_2} = \frac{p^2 a}{(p+q)(pa - qb)}.$$

How is the character of the telescope altered by the above arrangement?

58. If a string be wrapped round a glass prism whose section is an equilateral triangle, so as to be always inclined at the same angle to the axis of the prism, the portions of the string seen by internal reflexion will appear to be parallel to the portions seen directly.

59. If the Earth, supposed spherical, were covered to a depth h with water, h being small compared with r the

Earth's radius; shew that the height to which a person must be raised above the surface of the water in order to see as far below the horizon as when he was on the surface of the Earth is $\dfrac{h^2}{2r(\mu^2-1)}$ nearly, μ being the index of refraction for water.

60. The equation of the boundary of a comet or nebulous body being given, together with the density at any point of its interior, it is required to find the law of brightness of its apparent disk, supposing that each particle sends the same quantity of light to the eye of a distant spectator.

Ex. A spherical body of radius a, the density at any distance r from the centre being $\Delta\sqrt{(a^2-r^2)}$.

Conversely. From the law of brightness of the apparent disc, and the density at any point and known distance of the body, find the equation of the boundary,—supposing it to be symmetrical with reference to a plane perpendicular to the direction of vision.

Ex. The density of the body being the same as in the foregoing example, the brightness at any apparent distance R from the centre of the disk varies as $a^2-m^2R^2$.

61. In the phenomenon of Solar Halos how are the positions and colours of the inner and outer circles surrounding the Sun, and the positions of the Parhelia, theoretically explained? Why are the Parhelia not exactly coincident with the inner circle?

62. Within a reflecting circle on the same side of the centre are two parallel rays, one dividing the circumference into arcs which are as 3 : 1,—the other dividing it into arcs which are as 8 : 1; find the least value of n such that after each ray has suffered n reflexions, they may be again parallel.

63. Given the directions of three plane mirrors in space, construct a straight line such that if light from it be reflected by the three mirrors in succession, the third image shall be parallel to the straight line.

Walton and Mackenzie, *Solutions of the Cambridge Problems*, 1854, p. 73.

64. Rays are incident parallel to the axis of x on a reflecting curve, and the equation to the catacaustic is $y = f(x)$, the equation to the reflecting curve will result from eliminating t from the two equations

$$x - t = \frac{y - f(t)}{f'(t)} = \frac{C - \int [1 + \sqrt{\{1 + (f'(t))^2\}}]\, dt}{1 + \sqrt{[1 + \{f'(t)\}^2]}}.$$

Boole's *Differential Equations*, p. 257.

65. Supposing a very large number of hexagonal crystals of ice to descend continually in the atmosphere with their axes vertical and faces turned in all possible directions,—prove that the reflexions of the light of the sun or moon from the vertical faces will cause a spectator on the earth's surface to see a horizontal circle of white light of the same breadth and altitude as the luminary. What phenomenon may be explained by this result?

66. When a large number of crystals of ice of the form of hexagonal prisms with plane ends perpendicular to their axes are suspended in the air, it is found that many of the simple crystals are joined together at their ends or sides so that the axes are parallel. Hence shew that (i) those of the compound crystals which have the axes horizontal, and are situated opposite to the sun at the same altitude, send light to a spectator by two reflexions at vertical planes inclined at an angle of 90° to each other, whatever be the orientation of the planes; (ii) those which have the axes vertical, and are situated in an azimuth of 120° from the sun at the same altitude, send light to the spectator by reflexions at planes inclined to each other at an angle of 120°.

What phenomena observed *sometimes to accompany Solar Halos are explained by these results?*

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AT THE UNIVERSITY PRESS.

By the same Author.

ELEMENTARY TREATISE ON MECHANICS,

WITH A COLLECTION OF EXAMPLES.

285 pp. (1855) Crown 8vo., cloth, *9s. 6d.*

It has been endeavoured to render the volume suitable as a Manual for the junior classes in Universities and the higher classes in Schools. With this object have been included in it those portions of Theoretical Mechanics which can be conveniently investigated without the Differential Calculus, and with one or two short exceptions, the student is not presumed to require a knowledge of any branches of Mathematics beyond the elements of Algebra, Geometry, and Trigonometry. A Collection of Problems and Examples has been added chiefly taken from the Cambridge Senate-house and College Examination Papers, which will, it is trusted, be found useful as an exercise for the student.

MACMILLAN AND CO.

CAMBRIDGE, AND 23, HENRIETTA STREET, COVENT GARDEN, LONDON.

LIST OF BOOKS

PUBLISHED BY

MACMILLAN AND CO.

Cambridge,

AND 23, HENRIETTA STREET, COVENT GARDEN, LONDON.

ABDY.—A Historical Sketch of Civil Procedure among the Romans. By J. T. ABDY, LL.D., Regius Professor of Civil Law in the University of Cambridge. Crown 8vo. cloth, *4s. 6d.*

ÆSCHYLI Eumenides.

The Greek Text with English Notes, and an Introduction, containing an Analysis of Müller's Dissertations. By BERNARD DRAKE, M.A., late Fellow of King's College, Cambridge. 8vo. cloth, *7s. 6d.*

ADAMS.—The Twelve Foundations and other Poems.

By H. C. ADAMS, M.A., Author of "Sivan the Sleeper," &c. Royal 16mo. cloth, 5s.

AGNES HOPETOUN'S SCHOOLS AND HOLIDAYS.

The Experiences of a Little Girl. A Story for Girls. By Mrs. OLIPHANT, Author of "Margaret Maitland." Royal 16mo. 6s.

AIRY.—Mathematical Tracts on the Lunar and Planetary Theories. The Figure of the Earth, Precession and Nutation, the Calculus of Variations, and the Undulatory Theory of Optics. By G. B. AIRY, M.A., Astronomer Royal. **Fourth Edition,** revised and improved. 8vo. cloth, 15s.

ARISTOTLE on the Vital Principle.

Translated, with Notes. By CHARLES COLLIER, M.D., F.R.S., Fellow of the Royal College of Physicians. Crown 8vo. cloth, *8s. 6d.*

BEAMONT.—Catherine, the Egyptian Slave in 1852.

A Tale of Eastern Life. By W. J. BEAMONT, M.A., Fellow of Trinity College, Cambridge, late Principal of the English College, Jerusalem. Fcap. 8vo. *5s. 6d.*

BEASLEY.—An Elementary Treatise on Plane Trigonometry; with a numerous Collection of Examples. By R. D. BEASLEY, M.A., Fellow of St. John's College, Cambridge, Head-Master of Grantham Grammar School. Crown 8vo. cloth, *3s. 6d.*

BIRKS.—The Difficulties of Belief in connexion with the Creation and the Fall. By THOMAS RAWSON BIRKS, M.A., Rector of Kelshall, and EXAMINING CHAPLAIN TO THE LORD BISHOP OF CARLISLE; Author of "The Life of the Rev. E. Bickersteth." Crown 8vo. cloth, *4s. 6d.*

BLAKESLEY.—Four Months in Algeria.
With a Visit to Carthage. With Maps and Illustrations after Photographs. By J. W. BLAKESLEY, B.D., Vicar of Ware, and sometime Fellow and Tutor of Trinity College, Cambridge. 8vo. cloth, 14*s.*

BOLTON.—The Evidences of Christianity.
By W. J. BOLTON, of Caius College, Cambridge. 8vo. 6*s.*

BOOLE.—The Mathematical Analysis of Logic.
By GEORGE BOOLE, F.R.S. Professor of Mathematics, in the Queen's University, Ireland. 8vo. sewed, 5*s.*

BOOLE.—A Treatise on Differential Equations.
Crown 8vo. cloth, 14*s.*

BOWEN.—The Force of Habit considered as an Argument to prove the Moral Government of Man by God. By E. E. BOWEN, B.A., Scholar of Trinity College, Cambridge. Crown 8vo. 1*s.*

BRAVE WORDS for BRAVE SOLDIERS and SAILORS.
Tenth Thousand. 16mo. sewed, 2*d.*; or 10*s.* per 100.

BRETT.—Suggestions relative to the Restoration of Suffragan Bishops and Rural Deans. By THOMAS BRETT (A.D. 1711). Edited by JAMES FENDALL, M.A., Procter in Convocation for the Clergy of Ely. Crown 8vo. cloth, 2*s.* 6*d.*

BRIMLEY.—Essays, by the late GEORGE BRIMLEY, M.A.
Edited by W. G. CLARK, M.A., Tutor of Trinity College, and Public Orator in the University of Cambridge. With Portrait. Crown 8vo. cloth, 7*s.* 6*d.*

CONTENTS.—I. Tennyson's Poems.—II. Wordsworth's Poems.—III. Poetry and Criticism.—IV. Angel in the House.—V. Carlyle's Life of Stirling. VI. Esmond.—VII. My Novel.—VIII. Bleak House.—IX. Westward Ho!—X. Wilson's Noctes.—XI. Comte's Positive Philosophy.

BUTLER (Archer).—Sermons, Doctrinal and Practical.
By the Rev. WILLIAM ARCHER BUTLER, M.A. late Professor of Moral Philosophy in the University of Dublin. Edited, with a Memoir of the Author's Life, by the Very Rev. THOMAS WOODWARD, M.A. Dean of Down. With Portrait. **Fifth Edition.** 8vo. cloth, 12*s.*

BUTLER (Archer).—A Second Series of Sermons.
Edited by J. A. JEREMIE, D.D. Regius Professor of Divinity in the University of Cambridge. **Third Edition.** 8vo. cloth, 10*s.* 6*d.*

BUTLER (Archer).—History of Ancient Philosophy.
A Series of Lectures. Edited by WILLIAM HEPWORTH THOMPSON, M.A. Regius Professor of Greek in the University of Cambridge. 2 vols. 8vo. cloth, 1*l.* 5*s.*

BUTLER (Archer).—Letters on Romanism, in Reply to Mr. NEWMAN'S Essay on Development. Edited by the Very Rev. T. WOODWARD, Dean of Down. **Second Edition,** revised by the Ven. Archdeacon HARDWICK. 8vo. cloth, 10*s.* 6*d.*

CAMBRIDGE.—Cambridge Theological Papers. Comprising those given at the Voluntary Theological and Crosse Scholarship Examinations. Edited, with References and Indices, by A. P. MOOR, M.A. of Trinity College, Cambridge, and Sub-warden of St. Augustine's College, Canterbury. 8vo. cloth, 7*s.* 6*d.*

CAMBRIDGE SENATE-HOUSE PROBLEMS and RIDERS, with SOLUTIONS:—

1848—1851.—Problems.	By N. M. FERRERS, M.A. and J. S. JACKSON, M.A. of Caius College. 15*s*. 6*d*.
1848—1851.—Riders.	By F. J. JAMESON, M.A. of Caius College. 7*s*. 6*d*.
1854—Problems and Riders.	By W. WALTON, M.A. of Trinity College, and C. F. MACKENZIE, M.A. of Caius College. 10*s*. 6*d*.
1857—Problems and Riders.	By W. M. CAMPION, M.A. of Queen's College, and W. WALTON, M.A. of Trinity College. 8*s*. 6*d*.

CAMBRIDGE ENGLISH PRIZE POEMS, which have obtained the Chancellor's Gold Medal from the institution of the Prize to 1858. Crown 8vo. cloth, 7*s*. 6*d*.

CAMBRIDGE.—Cambridge and Dublin Mathematical Journal.
The Complete Work, in Nine Vols. 8vo. cloth, 7*l*. 4*s*.

ONLY A FEW COPIES OF THE COMPLETE WORK REMAIN ON HAND.

CAMPBELL.—The Nature of the Atonement and its Relation to Remission of Sins and Eternal Life. By JOHN MCLEOD CAMPBELL, formerly Minister of Row. 8vo. cloth, 10*s*. 6*d*.

CICERO.—Old Age and Friendship.
Translated into English. Two Parts. 12mo. sewed, 2*s*. 6*d*. each.

COLENSO.—The Colony of Natal. A Journal of Ten Weeks' Tour of Visitation among the Colonists and Zulu Kafirs of Natal. By the Right Rev. JOHN WILLIAM COLENSO, D.D. Lord Bishop of Natal, with a Map and Illustrations. Fcap. 8vo. cloth, 5*s*.

COLENSO.—Village Sermons.
Second Edition. Fcap. 8vo. cloth, 2*s*. 6*d*.

COLENSO.—Four Sermons on Ordination, and on Missions.
18mo. sewed, 1*s*.

COLENSO.—Companion to the Holy Communion, containing the Service, and Select Readings from the writings of Mr. MAURICE. Edited by the Lord Bishop of Natal. *Fine Edition*, rubricated and bound in morocco, antique style, 6*s*.; or in cloth, 2*s*. 6*d*. *Common Paper*, limp cloth, 1*s*.

COTTON.—Sermons and Addresses delivered in Marlborough College during Six Years by GEORGE EDWARD LYNCH COTTON, D.D., Lord Bishop of Calcutta, and Metropolitan of India. Crown 8vo. cloth, 10*s*. 6*d*.

COTTON.—Sermons: chiefly connected with Public Events of 1854. Fcap. 8vo. cloth, 3*s*.

CROSSE.—An Analysis of Paley's Evidences.
By C. H. CROSSE, M.A. of Caius College, Cambridge. 12mo. boards, 3*s*. 6*d*.

DAVIES.—St. Paul and Modern Thought:
Remarks on some of the Views advanced in Professor Jowett's Commentary on St. Paul. By Rev. J. LL. DAVIES, M.A. Fellow of Trinity College, Cambridge, and Rector of Christ Church, Marylebone. 8vo. sewed, 2*s*. 6*d*.

DAYS OF OLD: Stories from Old English History of the Druids, the Anglo-Saxons, and the Crusades. By the Author of "Ruth and her Friends." Imp. 16mo. cloth, 5*s.*

DEMOSTHENES.—Demosthenes de Corona. The Greek Text, with English Notes. By BERNARD DRAKE, M.A. late Fellow of King's Coll. Cambridge, Editor and Translator of the "Eumenides of Æschylus." Crown 8vo. cloth, 5*s.*

DEMOSTHENES.—Demosthenes on the Crown. Translated by J. P. NORRIS, M.A. Fellow of Trinity College, Cambridge, and one of Her Majesty's Inspectors of Schools. Crown 8vo. cloth, 3*s.*

DREW.—A Geometrical Treatise on Conic Sections, with Copious Examples from the Cambridge Senate House Papers. By W. H. DREW, M.A. of St. John's College, Cambridge, Second Master of Blackheath Proprietary School. Crown 8vo. cloth, 4*s.* 6*d.*

EVANS.—Sonnets on the Death of Wellington. By SEBASTIAN EVANS, of Emmanuel College, Cambridge. 8vo. sewed, 1*s.*

FARRAR.—The Influence of Classical Studies on English Literature in the Reigns of Elizabeth and James I. By F. W. FARRAR, Fellow of Trinity College, Cambridge. 8vo. sewed, 1*s.* 6*d.*

FARRAR.—Lyrics of Life. By FREDERIC W. FARRAR, Fellow of Trinity College, Cambridge. Author of "Eric," &c. Fcap. 8vo. cloth, 4*s.* 6*d.*

FORD.—Steps to the Sanctuary; or, the Order for Morning Prayer, set forth and explained in Verse. By JAMES FORD, M.A., Prebendary of Exeter Cathedral. Crown 8vo. cloth, 2*s.* 6*d.*

FROST.—The First Three Sections of Newton's Principia. With Notes and Problems in illustration of the subject. By PERCIVAL FROST, M.A. late Fellow of St. John's College, Cambridge, and Mathematical Lecturer of Jesus College. Crown 8vo. cloth, 10*s.* 6*d.*

GILL.—The Anniversaries. Poems in Commemoration of Great Men and Great Events. By T. H. GILL. Fcap. 8vo. cloth, 5*s.*

GODFRAY.—An Elementary Treatise on the Lunar Theory. With a brief Sketch of the History of the Problem up to the time of Newton. By HUGH GODFRAY, M.A. of St. John's College, Esquire Bedell in the University of Cambridge. 8vo. cloth, 5*s.* 6*d.*

GRANT.—Plane Astronomy. Including Explanations of Celestial Phenomena, and Descriptions of Astronomical Instruments. By A. R. GRANT, M.A., one of Her Majesty's Inspectors of Schools, late Fellow of Trinity College, Cambridge. 8vo. boards, 6*s.*

GRANT.—The Education of the Poor a Blessing not an Injury to the Nation. By the Rev. A. R. GRANT, M.A. Crown 8vo. 6*d.*

GRIFFIN.—The Theory of Double Refraction. By W. N. GRIFFIN, B.D., late Fellow of St. John's College, Cambridge. 8vo. 2*s.*

HALE.—The Transportation Question; or, Why Western Australia should be made a Reformatory Colony instead of a Penal Settlement. By the Right Rev. MATTHEW B. HALE, D.D. Lord Bishop of Perth, W.A. Crown 8vo. sewed 2*s.* 6*d.*

HALLIFAX.—Analysis of the Civil Law, being a comparison of the Roman and English Law. By BISHOP HALLIFAX, with Additions, by J. W. GELDART, LL.D. of Trinity Hall, Cambridge. 8vo. boards, 8*s*. 6*d*.

HAMILTON.—On Truth and Error: Thoughts, in Prose and Verse, on the Principles of Truth, and the Causes and Effects of Error. By JOHN HAMILTON, Esq. (of St. Ernan's), M.A. St. John's College, Cambridge. Crown 8vo. cloth, 10*s*. 6*d*.

HARE.—Charges delivered during the Years 1840 to 1854. With Notes on the Principal Events affecting the Church during that period. By JULIUS CHARLES HARE, M.A. sometime Archdeacon of Lewes, and Chaplain in Ordinary to the Queen. With an Introduction, explanatory of his position in the Church with reference to the parties which divide it, 3 vols. 8vo. cloth, 1*l*. 11*s*. 6*d*.

HARE.—Miscellaneous Pamphlets on some of the Leading Questions agitated in the Church during the Years 1845—51. 8vo. cloth, 12*s*.

HARE.—The Victory of Faith.
Second Edition. 8vo. cloth, 5*s*.

HARE.—The Mission of the Comforter.
Second Edition. With Notes. 8vo. cloth, 12*s*.

HARE.—Vindication of Luther from his English Assailants.
Second Edition. 8vo. cloth, 7*s*.

HARE.—Parish Sermons.
Second Series. 8vo. cloth, 12*s*.

HARE.—Sermons Preacht on Particular Occasions.
8vo. cloth, 12*s*.

⁂ The two following Books are included in the Three Volumes of Charges, and may still be had separately.

HARE.—The Contest with Rome.
With Notes, especially in answer to Dr. Newman's Lectures on Present Position of Catholics. **Second Edition.** 8vo. cloth, 10*s*. 6*d*.

HARE.—Charges delivered in the Years 1843, 1845, 1846.
Never before published. With an Introduction, explanatory of his position in the Church with reference to the parties which divide it. 6*s*. 6*d*.

HARE.—Portions of the Psalms in English Verse.
Selected for Public Worship. 18mo. cloth, 2*s*. 6*d*.

HARE.—Two Sermons preached in Herstmonceux Church, on Septuagesima Sunday, 1855, being the Sunday after the Funeral of the Venerable Archdeacon Hare. By the Rev. H. VENN ELLIOTT, Perpetual Curate of St. Mary's, Brighton, late Fellow of Trinity College, Cambridge, and the Rev. J. N. SIMPKINSON, Rector of Brington, Northampton, formerly Curate of Herstmonceux. 8vo. 1*s*. 6*d*.

HARDWICK.—Christ and other Masters.
A Historical Inquiry into some of the chief Parallelisms and Contrasts between Christianity and the Religious Systems of the Ancient World. With special reference to prevailing Difficulties and Objections. By the Ven. ARCHDEACON HARDWICK. Part I. INTRODUCTION. PART II. THE RELIGIONS OF INDIA. Part III. THE RELIGIONS OF CHINA, AMERICA AND OCEANICA. Part IV. RELIGIONS OF EGYPT AND MEDO-PERSIA. 8vo. cloth. 7*s*. 6*d*. each part.

HARDWICK.—A History of the Christian Church, during the Middle Ages and the Reformation. (A.D. 590–1600.)
By Archdeacon Hardwick. Two vols. crown 8vo. cloth, 21*s.*

Vol. I. History from Gregory the Great to the Excommunication of Luther. With Maps.

Vol. II. History of the Reformation of the Church.

Each volume may be had separately. Price 10*s.* 6*d.*

⁂ These Volumes form part of the Series of Theological Manuals.

HARDWICK.—Twenty Sermons for Town Congregations.
Crown 8vo. cloth, 6*s.* 6*d.*

HAYNES.—Outlines of Equity. By FREEMAN OLIVER HAYNES, Barrister-at-Law, late Fellow of Caius College, Cambridge. Crown 8vo. cloth, 10*s.*

HEDDERWICK.—Lays of Middle Age, and other Poems.
By JAMES HEDDERWICK. Fcp. 8vo. 5*s.*

HEMMING.—An Elementary Treatise on the Differential and Integral Calculus. By G. W. HEMMING, M.A. Fellow of St. John's College, Cambridge. **Second Edition.** 8vo. cloth, 9*s.*

HERVEY.—The Genealogies of our Lord and Saviour Jesus Christ, as contained in the Gospels of St. Matthew and St. Luke, reconciled with each other and with the Genealogy of the House of David, from Adam to the close of the Canon of the Old Testament, and shown to be in harmony with the true Chronology of the Times. By Lord ARTHUR HERVEY, M.A. Rector of Ickworth. 8vo. cloth, 10*s.* 6*d.*

HERVEY.—The Inspiration of Holy Scripture.
Five Sermons preached before the University of Cambridge. 8vo. cloth, 3*s.* 6*d.*

HOWARD.—The Pentateuch; or, the Five Books of Moses.
Translated into English from the Version of the LXX. With Notes on its Omissions and Insertions, and also on the Passages in which it differs from the Authorised Version. By the Hon. HENRY HOWARD, D.D. Dean of Lichfield. Crown 8vo. cloth. GENESIS, 1 vol. 8*s.* 6*d.*; EXODUS AND LEVITICUS, 1 vol. 10*s.* 6*d.*; NUMBERS AND DEUTERONOMY, 1 vol. 10*s.* 6*d.*

HUMPHRY.—The Human Skeleton (including the Joints).
By GEORGE MURRAY HUMPHRY, M.D. Cantab. F.R.C.S., Surgeon to Addenbrooke's Hospital, Lecturer on Surgery and Anatomy in the Cambridge University Medical School. With Two Hundred and Sixty Illustrations drawn from Nature. Medium 8vo. cloth, 1*l.* 8*s.*

INGLEBY.—Outlines of Theoretical Logic.
Founded on the New Analytic of SIR WILLIAM HAMILTON. Designed for a Text-book in Schools and Colleges. By C. MANSFIELD INGLEBY, M.A of Trinity College, Cambridge. In fcap. 8vo. cloth, 3*s.* 6*d.*

JAMESON.—Analogy between the Miracles and Doctrines of Scripture. By F. J. JAMESON, M.A., Fellow of St. Catharine's College, Cambridge. Fcap. 8vo. cloth, 2*s.*

JAMESON.—Brotherly Counsels to Students. Four Sermons preached in the Chapel of St. Catharine's College, Cambridge. By F. J JAMESON, M.A. Fcap. 8vo. limp cloth, red edges, 1*s.* 6*d.*

JEWELL.—An Apology of the Church of England, and an Epistle to Seignior Scipio concerning the Council of Trent, translated from the original Latin, and illustrated with Notes, chiefly drawn from the Author's "Defence of the Apology." By A. T. RUSSELL. Fcp. 8vo. bds. 5*s.*

JUSTIN MARTYR.—S. Justini Philosophi et Martyris Apologia Prima. Edited, with a corrected Text, and English Introduction with explanatory Notes, by W. TROLLOPE, M.A. Pembroke College, Cambridge. 8vo. bds. 7*s.* 6*d.*

JUVENAL.—Juvenal, for Schools.
With English Notes. By J. E. B. MAYOR, M.A. Fellow and Assistant Tutor of St. John's College, Cambridge. Crown 8vo. cloth, 10*s.* 6*d.*

KENNEDY.—The Influence of Christianity on International Law. By C. M. KENNEDY, of Caius College, Cambridge. Crown 8vo. cloth, 4*s.*

KINGSLEY.—Two Years Ago.
By CHARLES KINGSLEY, F.S.A. Rector of Eversley, and Chaplain in Ordinary to the Queen. **Second Edition.** 3 vols. crown 8vo. cloth, 1*l.* 11*s.* 6*d.*

KINGSLEY.—"Westward Ho!" or, the Voyages and Adventures of Sir Amyas Leigh, Knight of Burrough, in the County of Devon, in the Reign of Her Most Glorious Majesty Queen Elizabeth. **Third Edition.** Crown 8vo. cloth, 7*s.* 6*d.*

KINGSLEY.—Glaucus; or, the Wonders of the Shore.
New and Illustrated Edition, corrected and enlarged. Containing beautifully Coloured Illustrations of the Objects mentioned in the Work. Elegantly bound in cloth, with gilt leaves. 7*s.* 6*d.*

KINGSLEY.—The Heroes: or, Greek Fairy Tales for my Children. With Eight Illustrations, Engraved by WHYMPER. **New Edition,** printed on toned paper, and elegantly bound in cloth, with gilt leaves, Imp. 16mo. 5*s.*

KINGSLEY.—Alexandria and Her Schools: being Four Lectures delivered at the Philosophical Institution, Edinburgh. With a Preface. Crown 8vo. cloth, 5*s.*

KINGSLEY.—Phaethon; or Loose Thoughts for Loose Thinkers. **Third Edition.** Crown 8vo. boards, 2*s.*

KINGSLEY.—The Recollections of Geoffry Hamlyn.
By HENRY KINGSLEY, Esq. 3 Vols. 1*l.* 11*s.* 6*d.*

LATHAM.—The Construction of Wrought-Iron Bridges, embracing the Practical Application of the Principles of Mechanics to Wrought-Iron Girder Work. By J. H. LATHAM, Esq. Civil Engineer. 8vo. cloth. With numerous detail Plates. 15*s.*

LECTURES TO LADIES ON PRACTICAL SUBJECTS. **Third Edition,** revised. Crown 8vo. cloth, 7*s.* 6*d.* By Reverends F. D. MAURICE, CHARLES KINGSLEY, J. LL. DAVIES, ARCHDEACON ALLEN, DEAN TRENCH, PROFESSOR BREWER, DR. GEORGE JOHNSON, DR. SIEVEKING, DR. CHAMBERS, F. J. STEPHEN, Esq., and TOM TAYLOR, Esq.

CONTENTS:—Plan of Female Colleges—The College and the Hospital—The Country Parish — Over Work and Anxiety — Dispensaries — District Visiting—Influence of Occupation on Health—Law as it affects the Poor—Everyday Work of Ladies—Teaching by Words—Sanitary Law—Workhouse Visiting.

LUDLOW.—British India; its Races, and its History, down to 1857. By JOHN MALCOLM LUDLOW, Barrister-at-Law. 2 vols. fcap. 8vo. cloth, 9*s.*

LUSHINGTON.—La Nation Boutiquière: and other Poems, chiefly Political. With a Preface. By the late HENRY LUSHINGTON, Chief Secretary to the Government of Malta. **Points of War.** By FRANKLIN LUSHINGTON, Judge in the Supreme Courts of the Ionian Isles. In 1 vol. fcap 8vo. cloth, 3*s.*

LUSHINGTON.—The Italian War 1848-9, and the Last Italian Poet. By the late HENRY LUSHINGTON, Chief Secretary to the Government of Malta. With a Biographical Preface by G. S. VENABLES. Crown 8vo. cloth, 6*s.* 6*d.*

MACKENZIE.—The Christian Clergy of the first Ten Centuries, and their Influence on European Civilization. By HENRY MACKENZIE, B.A. Scholar of Trinity College, Cambridge. Crown 8vo. cloth, 6*s.* 6*d.*

MACLEAR.—The Christian Statesman and Our Indian Empire. By G. F. MACLEAR, B.A. Curate of St. Barnabas, Kennington. **Second Edition.** Crown 8vo. cloth, 3*s.* 6*d.*

MANSFIELD.—Paraguay, Brazil, and the Plate. With a Map, and numerous Woodcuts. By CHARLES MANSFIELD, M.A. of Clare College, Cambridge. With a Sketch of his Life. By the Rev. CHARLES KINGSLEY. Crown 8vo. cloth, 12*s.* 6*d.*

M'COY.—Contributions to British Palæontology; or, First Descriptions of several hundred Fossil Radiata, Articulata, Mollusca, and Pisces from the Tertiary, Cretaceous, Oolitic, and Palæozoic Strata of Great Britain. With numerous Woodcuts. By FREDERICK MCCOY, F.G.S., Professor of Natural History in the University of Melbourne. 8vo. cloth, 9*s.*

MASSON.—Essays, Biographical and Critical; chiefly on the English Poets. By DAVID MASSON, M.A. Professor of English Literature in University College, London. 8vo. cloth, 12*s.* 6*d.*

CONTENTS:—I. Shakespeare and Goethe.—II. Milton's Youth.—III. The Three Devils: Luther's, Milton's, and Goethe's.—IV. Dryden, and the Literature of the Restoration.—V. Dean Swift.—VI. Chatterton: a Story of the Year 1770.—VII. Wordsworth.—VIII. Scottish Influence on British Literature.—IX. Theories of Poetry.—X. Prose and Verse: De Quincey.

MASSON.—British Novelists and their Styles; being a Critical Sketch of the History of British Prose Fiction. By DAVID MASSON, M.A. Crown 8vo. cloth, *7s. 6d.*

MASSON.—Life of John Milton, narrated in Connexion with the Political, Ecclesiastical, and Literary History of his Time. Vol. I. with Portraits. 18*s.*

MAURICE.—Expository Works on the Holy Scriptures.
By FREDERICK DENISON MAURICE, M.A., Chaplain of Lincoln's Inn.

I.—The Patriarchs and Lawgivers of the Old Testament.
Second Edition. Crown 8vo. cloth, 6*s.*

This volume contains Discourses on the Pentateuch, Joshua, Judges, and the beginning of the First Book of Samuel.

II.—The Prophets and Kings of the Old Testament.
Second Edition. Crown 8vo. cloth, 10*s.* *6d.*

This volume contains Discourses on Samuel I. and II., Kings I. and II., Amos, Joel, Hosea, Isaiah, Micah, Nahum, Habakkuk, Jeremiah, and Ezekiel.

III.—The Unity of the New Testament.
8vo. cloth, 14*s.*

This Volume contains Discourses on the Gospels of St. Matthew, St. Mark, and St. Luke; the Acts of the Apostles; the Epistles of St. James, St. Jude, St. Peter, and St. Paul.

IV.—The Gospel of St. John; a Series of Discourses.
Second Edition. Crown 8vo. cloth, 10*s.* *6d.*

V.—The Epistles of St. John; a Series of Lectures on Christian Ethics. Crown 8vo. cloth, *7s. 6d.*

MAURICE.—Expository Works on the Prayer-Book.

I.—The Ordinary Services.
Second Edition. Fcap. 8vo. cloth, *5s. 6d.*

II.—The Church a Family. Twelve Sermons on the Occasional Services. Fcap. 8vo. cloth, *4s. 6d.*

MAURICE.—What is Revelation? A Series of Sermons on the Epiphany; to which are added Letters to a Theological Student on the Bampton Lectures of Mr. MANSEL. Crown 8vo. cloth, 10*s.* *6d.*

MAURICE.—Lectures on Ecclesiastical History.
8vo. cloth, 10*s.* *6d.*

MAURICE.—Theological Essays.
Second Edition, with a new Preface and other additions. Crown 8vo. cloth, 10*s.* *6d.*

MAURICE.—The Doctrine of Sacrifice deduced from the Scriptures. With a Dedicatory Letter to the Young Men's Christian Association. Crown 8vo. cloth, *7s. 6d.*

MAURICE.—Christmas Day, and other Sermons.
8vo. cloth, 10*s.* *6d.*

MAURICE.—The Religions of the World, and their Relations to Christianity. **Third Edition.** Fcap. 8vo. cloth, 5*s.*

MAURICE.—On the Lord's Prayer.
Third Edition. Fcap. 8vo. cloth, 2*s.* 6*d.*

MAURICE.—On the Sabbath Day: the Character of the Warrior; and on the Interpretation of History. Fcap. 8vo. cloth, 2*s.* 6*d.*

MAURICE.—Learning and Working.—Six Lectures on the Foundation of Colleges for Working Men, delivered in Willis's Rooms, London, in June and July, 1854. Crown 8vo. cloth, 5*s.*

MAURICE.—The Indian Crisis. Five Sermons.
Crown 8vo. cloth, 2*s.* 6*d.*

MAURICE.—Law's Remarks on the Fable of the Bees.
Edited, with an Introduction of Eighty Pages, by FREDERICK DENISON MAURICE, M.A. Chaplain of Lincoln's Inn. Fcp. 8vo. cloth, 4*s.* 6*d.*

MAURICE.—Miscellaneous Pamphlets:—

- **I.—Eternal Life and Eternal Death.**
  Crown 8vo. sewed, 1*s.* 6*d.*
- **II.—Death and Life. A Sermon.** In Memoriam C. B. M.
  8vo. sewed, 1*s.*
- **III.—Plan of a Female College for the Help of the Rich** and of the Poor. 8vo. 6*d.*
- **IV.—Administrative Reform.**
  Crown 8vo. 3*d.*
- **V.—The Word "Eternal," and the Punishment of the** Wicked. **Fifth Thousand.** 8vo. 1*s.*
- **VI.—The Name "Protestant:" and the English Bishopric** at Jerusalem. **Second Edition.** 8vo. 3*s.*
- **VII.—Right and Wrong Methods of Supporting Protestantism.** 8vo. 1*s.*
- **VIII.—Thoughts on the Oxford Election of 1847.**
  8vo. 1*s.*
- **IX.—The Case of Queen's College, London.**
  8vo. 1*s.* 6*d.*
- **X.—The Worship of the Church a Witness for the** Redemption of the World. 8vo. sewed, 1*s.*

MAYOR.—Cambridge in the Seventeenth Century.
2 vols. fcap. 8vo. cloth, 13*s.*
Vol. I. Lives of Nicholas Ferrar.
Vol. II. Autobiography of Matthew Robinson.
By JOHN E. B. MAYOR, M.A. Fellow and Assistant Tutor of St. John's College, Cambridge.

⁂ The Autobiography of Matthew Robinson may be had separately, price 5*s.* 6*d.*

MAYOR.—Early Statutes of St. John's College, Cambridge.
Now first edited with Notes. Royal 8vo. 18*s.*

⁂ The First Part is now ready for delivery.

MAXWELL.—The Stability of the Motion of Saturn's Rings. By J. C. MAXWELL, M.A. Professor of Natural Philosophy in the University of Aberdeen. 4to. sewed, 6*s.*

MOORE.—A New Proof of the Method of Algebra commonly called "Greatest Common Measure." By B. T. MOORE, B.A., Fellow of Pembroke College, Cambridge. Crown 8vo. 6*d.*

MORGAN.—A Collection of Mathematical Problems and Examples. Arranged in the Different Subjects progressively, with Answers to all the Questions. By H. A. MORGAN, M.A., Fellow of Jesus College. Crown 8vo. cloth, 6*s.* 6*d.*

MORSE.—Working for God; Four Sermons Preached before the University of Cambridge. By FRANCIS MORSE, M.A. Incumbent of St. John's, Ladywood, Birmingham. Crown 8vo. sewed, 2*s.* 6*d.*

NAPIER.—Lord Bacon and Sir Walter Raleigh. Critical and Biographical Essays. By MACVEY NAPIER, late Editor of the *Edinburgh Review* and of the *Encyclopædia Britannica.* Post 8vo. cloth, 7*s.* 6*d.*

NORRIS.—Ten School-Room Addresses. Edited by J. P. NORRIS, M.A. Fellow of Trinity College, and one of Her Majesty's Inspectors of Schools. 18mo. sewed, 8*d.*

NORWAY AND SWEDEN.—A Long Vacation Ramble in 1856. By X and Y. Crown 8vo. cloth, 6*s.* 6*d.*

OCCASIONAL PAPERS on UNIVERSITY and SCHOOL MATTERS; containing an Account of all recent University Subjects and Changes. Two Parts are now ready, price 1*s.* each.

OLIPHANT.—Agnes Hopetoun's Schools and Holidays. The Experiences of a Little Girl. By Mrs. OLIPHANT, Author of "Margaret Maitland." Royal 16mo. cloth, 6*s.*

PARKINSON.—A Treatise on Elementary Mechanics. For the Use of the Junior Classes at the University, and the Higher Classes in Schools. With a Collection of Examples. By S. PARKINSON, B.D. Fellow and Assistant Tutor of St. John's College, Cambridge. Crown 8vo. cloth, 9*s.* 6*d.*

PARMINTER.—Materials for a Grammar of the Modern English Language. Designed as a Text-book of Classical Grammar for the use of Training Colleges, and the Higher Classes of English Schools. By GEORGE HENRY PARMINTER, of Trinity College, Cambridge; Rector of the United Parishes of SS. John and George, Exeter. Fcap. 8vo. cloth, 3*s.* 6*d.*

PEROWNE.—"Al-Adjrumiieh." An Elementary Arabic Grammar. By J. J. S. PEROWNE, B.D. Lecturer in Divinity in King's College, London, and Examining Chaplain to the Lord Bishop of Norwich. 8vo. cloth, 5*s.*

PERRY.—Sermons Preached before the University of Cambridge. By the Right Rev. CHARLES PERRY, Lord Bishop of Melbourne, formerly Fellow and Tutor of Trinity College, Cambridge. Crown 8vo. cloth, 3*s.*

PHEAR.—Elementary Hydrostatics.
By J. B. Phear, M.A. Fellow of Clare College, Cambridge. **Second Edition.** Accompanied by numerous Examples, with the Solutions. Crown 8vo. cloth, *5s. 6d.*

PLAIN RULES ON REGISTRATION OF BIRTHS AND DEATHS. Crown 8vo. sewed, *1d.*; *9d.* per dozen; *5s.* per 100.

PLATO.—The Republic of Plato.
Translated into English, with Notes. By Two Fellows of Trinity College, Cambridge, (J. Ll. Davies M.A., and D. J. Vaughan, M.A.) **Second Edition.** 8vo. cloth, *10s. 6d.*

PRINCIPLES of ETHICS according to the NEW TESTAMENT. Crown 8vo. sewed, *2s.*

PROCTER.—A History of the Book of Common Prayer: with a Rationale of its Offices. By FRANCIS PROCTER, M.A., Vicar of Witton, Norfolk, and late Fellow of St. Catherine's College. **Third Edition,** revised and enlarged. Crown 8vo. cloth, *10s. 6d.*

*** This forms part of the Series of Theological Manuals.

PUCKLE.—An Elementary Treatise on Conic Sections and Algebraic Geometry. With a numerous collection of Easy Examples progressively arranged, especially designed for the use of Schools and Beginners. By G. HALE PUCKLE, M.A., Principal of Windermere College. **Second Edition,** enlarged and improved. Crown 8vo. cloth, *7s. 6d.*

PURTON.—Corporate Life.
A Sermon Preached before the University of Cambridge. By J. S. PURTON, B.D. Fellow and Tutor of St. Catherine's College. 8vo. sewed, *1s.*

RAMSAY.—The Catechiser's Manual; or, the Church Catechism illustrated and explained, for the use of Clergymen, Schoolmasters, and Teachers. By ARTHUR RAMSAY, M.A. of Trinity College, Cambridge. 18mo. cloth, *3s. 6d.*

REICHEL.—The Lord's Prayer and other Sermons.
By C. P. REICHEL, B.D., Professor of Latin in the Queen's University; Chaplain to his Excellency the Lord-Lieutenant of Ireland; and late Donnellan Lecturer in the University of Dublin. Crown 8vo. cloth, *7s. 6d.*

ROBINSON.—Missions urged upon the State, on Grounds both of Duty and Policy. By C. K. ROBINSON, M.A. Fellow and Assistant Tutor of St. Catherine's College. Fcap. 8vo. cloth, *3s.*

RUTH AND HER FRIENDS. A Story for Girls.
With a Frontispiece. **Second Edition.** Fcap. 8vo. cloth, *5s.*

SALLUST.—Sallust for Schools.
With English Notes. **Second Edition.** By CHARLES MERIVALE, B.D.; late Fellow and Tutor of St. John's College, Cambridge, &c., Author of the "History of Rome," &c. Fcap. 8vo. cloth, *4s. 6d.*

"THE JUGURTHA" AND "THE CATILINA" MAY BE HAD SEPARATELY, price *2s. 6d.* EACH IN CLOTH.

SCOURING OF THE WHITE HORSE; or, The Long Vacation Ramble of a London Clerk. By the Author of "Tom Brown's School Days." Illustrated by DOYLE. **Eighth Thousand.** Imp. 16mo. cloth, elegant, *8s. 6d.*

SELWYN.—The Work of Christ in the World.
Sermons preached before the University of Cambridge. By the Right Rev. GEORGE AUGUSTUS SELWYN, D.D. Bishop of New Zealand, formerly Fellow of St John's College. **Third Edition.** Crown 8vo. 2*s.*

SELWYN.—A Verbal Analysis of the Holy Bible.
Intended to facilitate the translation of the Holy Scriptures into Foreign Languages. Compiled for the use of the Melanesian Mission. Small folio, cloth, 14*s.*

SIMPSON.—An Epitome of the History of the Christian Church during the first Three Centuries and during the Reformation. With Examination Papers. By WILLIAM SIMPSON, M.A. **Third Edition.** Fcp. 8vo. cloth, 5*s.*

SMITH.—City Poems.
By ALEXANDER SMITH, Author of "A Life Drama," and other Poems. Fcap. 8vo. cloth. 5*s.*

SMITH.—Arithmetic and Algebra, in their Principles and Application: with numerous systematically arranged Examples, taken from the Cambridge Examination Papers. By BARNARD SMITH, M.A., Fellow of St. Peter's College, Cambridge. **Sixth Edition.** Crown 8vo. cloth, 10*s.* 6*d.*

SMITH.—Arithmetic for the use of Schools.
New Edition. Crown 8vo. cloth, 4*s.* 6*d.*

SMITH.—A Key to the Arithmetic for Schools.
Crown 8vo. cloth, 8*s.* 6*d.*

SMITH.—The Koran in India.
By LUMLEY SMITH, Fellow of Trinity Hall, Cambridge. 8vo. sewed, 2*s.*

SNOWBALL.—The Elements of Plane and Spherical Trigonometry. By J. C. SNOWBALL, M.A. Fellow of St. John's College, Cambridge. **Ninth Edition.** Crown 8vo. cloth, 7*s.* 6*d.*

SNOWBALL.—Introduction to the Elements of Plane Trigonometry for the use of Schools. **Second Edition.** 8vo. sewed, 5*s.*

SNOWBALL.—The Cambridge Course of Elementary Mechanics and Hydrostatics. Adapted for the use of Colleges and Schools. With numerous Examples and Problems. **Fourth Edition.** Crown 8vo. cloth, 5*s.*

SPRAY.
Crown 8vo. boards, 3*s.*

SWAINSON.—A Handbook to Butler's Analogy.
By C. A. SWAINSON, M.A. Principal of the Theological College, and Prebendary of Chichester. Crown 8vo. sewed, 2*s.*

SWAINSON.—The Creeds of the Church in their Relations to Holy Scripture and the Conscience of the Christian. 8vo. cloth, 9*s.*

CONTENTS:—I. Faith in God.—II. Exercise of our Reason.—III. Origin and Authority of Creeds.—IV. Inductive Proof of the Creeds.—V. Continual Guidance of the Spirit.—VI. Test and Application of Scripture.—VII. Private Judgment.—VIII. Strengthening of the Judgment and the Preparation for Controversy. With an Appendix.

TAIT and STEELE.—A Treatise on Dynamics, with numerous Examples. By P. G. TAIT, Fellow of St. Peter's College, Cambridge, and Professor of Mathematics in Queen's College, Belfast, and W. J. STEELE, late Fellow of St. Peter's College. Crown 8vo. cloth, 10*s*. 6*d*

TAYLOR.—The Restoration of Belief.

By ISAAC TAYLOR, Esq., Author of "The Natural History of Enthusiasm." Crown 8vo. cloth, 8*s*. 6*d*.

THEOLOGICAL Manuals.

CHURCH HISTORY: DURING THE MIDDLE AGES AND THE REFORMATION (A.D. 590—1600). By ARCHDEACON HARDWICK. With Four Maps, 2 vols. Crown 8vo. cloth, price 10*s*. 6*d*. each.

THE COMMON PRAYER: ITS HISTORY AND RATIONALE. By FRANCIS PROCTER. **Third Edition.** Crown 8vo. cloth, 10*s*. 6*d*.

HISTORY OF THE CANON OF THE NEW TESTAMENT. By B. F. WESTCOTT. Crown 8vo. cloth, 12*s*. 6*d*.

⁂ Others are in progress, and will be announced in due time.

THRING.—A Construing Book.

Compiled by the Rev. EDWARD THRING, M.A. Head Master of Uppingham Grammar School, late Fellow of King's College, Cambridge. Fcap. 8vo. cloth, 2*s*. 6*d*.

THRING.—The Elements of Grammar taught in English.

Second Edition. 18mo. bound in cloth, 2*s*.

THRING.—The Child's Grammar.

Being the substance of the above, with Examples for Practice. Adapted for Junior Classes. **A New Edition.** 18mo. limp cloth, 1*s*.

THRING.—Sermons delivered at Uppingham School.

Crown 8vo. cloth, 5*s*.

THRING.—School Songs.

A Collection of Songs for Schools. With the Music arranged for four Voices. Edited by EDWARD THRING, M.A., Head Master of Uppingham School, and H. RICCIUS. Small folio, 7*s*. 6*d*.

THRUPP.—Antient Jerusalem: a New Investigation into the History, Topography, and Plan of the City, Environs, and Temple. Designed principally to illustrate the records and prophecies of Scripture. With Map and Plans. By JOSEPH FRANCIS THRUPP, M.A. Vicar of Barrington, Cambridge, late Fellow of Trinity College. 8vo. cloth, 15*s*.

THOMPSON.—A Sermon preached after the Funeral of the late DEAN PEACOCK. By W. H. THOMPSON, M.A., Regius Professor of Greek in the University of Cambridge. 8vo. 1*s*.

THUCYDIDES, BOOK VI. With English Notes, and a Map. By PERCIVAL FROST, Jun. M.A. late Fellow of St. John's College, Cambridge. 8vo. 7*s*. 6*d*.

TODHUNTER.—A Treatise on the Differential Calculus. With numerous Examples. By I. TODHUNTER, M.A., Fellow and Assistant Tutor of St. John's College, Cambridge. **Second Edition,** Crown 8vo. cloth, 10*s*. 6*d*.

TODHUNTER.—A Treatise on the Integral Calculus.
With numerous Examples. Crown 8vo. cloth, 10*s.* 6*d.*

TODHUNTER.—A Treatise on Analytical Statics, with numerous Examples. **Second Edition.** Crown 8vo. cloth, 10*s.* 6*d.*

TODHUNTER.—A Treatise on Conic Sections, with numerous Examples. **Second Edition.** Crown 8vo. cloth, 10*s.* 6*d.*

TODHUNTER.—Algebra for the use of Colleges and Schools.
Crown 8vo. cloth, 7*s.* 6*d.*

TODHUNTER.—Plane Trigonometry for Colleges and Schools. Crown 8vo. cloth, 5*s.*

TODHUNTER.—A Treatise on Spherical Trigonometry for the Use of Colleges and Schools. Crown 8vo. cloth, 4*s.* 6*d.*

TODHUNTER.—Examples of Analytical Geometry of Three Dimensions. Crown 8vo. cloth, 4*s.*

TOM BROWN'S SCHOOL DAYS.
By AN OLD BOY. **Sixth Edition.** Crown 8vo. cloth, 10*s.* 6*d.*

TRENCH.—Synonyms of the New Testament.
By The Very Rev. RICHARD CHENEVIX TRENCH, D.D. Dean of Westminster. **Fourth Edition.** Fcap. 8vo. cloth, 5*s.*

TRENCH.—Hulsean Lectures for 1845—46.
CONTENTS. 1.—The Fitness of Holy Scripture for unfolding the Spiritual Life of Man. 2.—Christ the Desire of all Nations; or the Unconscious Prophecies of Heathendom. **Fourth Edition.** Foolscap 8vo. cloth, 5*s.*

TRENCH.—Sermons Preached before the University of Cambridge. Fcap. 8vo. cloth, 2*s.* 6*d.*

VAUGHAN.—Notes for Lectures on Confirmation. With suitable Prayers. By C. J. VAUGHAN, D.D., Head Master of Harrow School. **Second Edition.** Limp cloth, red edges, 1*s.* 6*d.*

VAUGHAN.—Sermons preached in St. John's Church, Leicester, during the years 1855 and 1856. By DAVID J. VAUGHAN, M.A. Fellow of Trinity College, Cambridge, and Incumbent of St. Mark's, Whitechapel. Crown 8vo. cloth, 5*s.* 6*d.*

VAUGHAN.—Three Sermons on The Atonement. With a Preface. By D. J. Vaughan, M.A. Limp cloth, red edges, 1*s.* 6*d.*

WAGNER.—Memoir of the Rev. George Wagner, late of St. Stephen's, Brighton. By J. N. SIMPKINSON, M.A. Rector of Brington, Northampton. **Second Edition.** Crown 8vo. cloth, 9*s.*

WATERS OF COMFORT.—A Small Volume of Devotional Poetry of a Practical Character. By the Author of "Visiting my Relations." Fcap. 8vo. cloth, 4*s.*

WESTCOTT.—History of the Canon of the New Testament during the First Four Centuries. By BROOKE FOSS WESTCOTT, M.A., Assistant Master of Harrow School; late Fellow of Trinity College, Cambridge. Crown 8vo. cloth, 12*s.* 6*d.*

*** **This forms part of the Series of Theological Manuals.**

WESTCOTT.—Characteristics of the Gospel Miracles.
Sermons preached before the University of Cambridge. **With Notes.** By B. F. WESTCOTT, M.A., Author of "History of the New Testament Canon." Crown 8vo. cloth, 4*s*. 6*d*.

WHITMORE.—Gilbert Marlowe and Other Poems.
With a Preface by the Author of "Tom Brown's Schooldays." Fcap. 8vo. cloth, 3*s*. 6*d*.

WILSON.—The Five Gateways of Knowledge.
By GEORGE WILSON, M.D., F.R.S.E., Regius Professor of Technology in the University of Edinburgh. **Second Edition.** Fcap. 8vo. cloth, 2*s*. 6*d*. or in Paper Covers, 1*s*.

WILSON.—The Progress of the Telegraph.
Fcap. 8vo. 1*s*.

WILSON.—A Treatise on Dynamics.
By W. P. WILSON, M.A., Fellow of St. John's, Cambridge, and Professor of Mathematics in the University of Melbourne. 8vo. bds. 9*s*. 6*d*.

WORSHIP OF GOD AND FELLOWSHIP AMONG MEN.
A Series of Sermons on Public Worship. Fcap. 8vo. cloth, 3*s*. 6*d*.

By F. D. MAURICE, M.A. T. J. ROWSELL, M.A. J. LL. DAVIES, M.A. and D. J. VAUGHAN, M.A.

WRIGHT.—The Iliad of Homer.
Translated into English Verse by J. C. WRIGHT, M.A. Translator of Dante. Crown 8vo. Books I.—VI. 5*s*.

WRIGHT.—Hellenica; or, a History of Greece in Greek,
as related by Diodorus and Thucydides, being a First Greek Reading Book, with Explanatory Notes, Critical and Historical. By J. WRIGHT, M.A., of Trinity College, Cambridge, and Head-Master of Sutton Coldfield Grammar School. **Second Edition,** WITH A VOCABULARY. 12mo. cloth, 3*s*. 6*d*.

WRIGHT.—A Help to Latin Grammar;
or, the Form and Use of Words in Latin. With Progressive Exercises. Crown 8vo. cloth, 4*s*. 6*d*.

WRIGHT.—The Seven Kings of Rome:
An easy Narrative, abridged from the First Book of Livy by the omission of difficult passages, being a First Latin Reading Book, with Grammatical Notes. Fcap. 8vo. cloth, 3*s*.

WRIGHT.—A Vocabulary and Exercises on the "Seven
Kings of Rome." Fcap. 8vo. cloth, 2*s*. 6*d*.

⁂ The Vocabulary and Exercises may also be had bound up with "The Seven Kings of Rome." Price 5*s*. cloth.

THE JOURNAL
OF
CLASSICAL AND SACRED PHILOLOGY.

No. XI. 4*s*.

Vols. I. to III. are now ready, 12*s*. 6*d*. each, cloth lettered.

CASES CAN BE HAD FOR BINDING THE VOLUMES.

R. CLAY, PRINTER, BREAD STREET HILL.

114 9.C.
[illegible]